J. Inv. Ill-Posed Problems **18** (2011), 935–944
DOI 10.1515/JIIP.2011.012

The optimum of the M. M. Lavrent'ev method

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Abstract. M. M. Lavrent'ev method for solving operator equation of the first kind in the Hilbert spaces have been consider. Exact estimates of error for the M. M. Lavrent'ev method have been make. The optimum of the M. M. Lavrent'ev method have been prove.

Keywords. Ill-posed problem, operator equation, regularization method, optimal, approximation, exact estimation.

1 Setting of the problem

Let U, F, V are Hilbert spaces, A is a linear bounded injective operator mapping U into F, having non-bounded inverse operator, B is a linear bounded operator mapping V into U, $M_r = B\overline{S}_r$, where $\overline{S}_r = \{v : v \in V, \|v\| \le r\}$, $N_r = AM_r$.

Assume that narrow $A^{-1}_{N_r}$ of the A^{-1} to N_r is uniformly continuous.

Consider operator equation of the first kind

$$Au = f; \quad u \in U,\ f \in F. \tag{1.1}$$

Assume that for $f = f_0$ there exists an exact solution u_0 of the equation (1.1) such that $u_0 \in M_r$. Instead of f_0 such $f_\delta \in F$ is given that $\|f_0 - f_\delta\| \le \delta$. Number $\delta > 0$ is given.

It is required by M_r, f_δ, δ to find an approximate solution u_δ of equation (1.1) and to evaluate $\|u_0 - u_\delta\|$ on the class M_r.

Further, by $C[F, U]$ we shall denote the space of the continuous operators mapping F into U.

We may introduce the quantitative characteristic $\Delta_\delta^{\text{opt}}$ using formula.

$$\Delta_\delta^{\text{opt}} = \inf\{\Delta_\delta(P) : P \in C[F, U]\}, \tag{1.2}$$

where

$$\Delta_\delta(P) = \sup_{u, f_\delta}\{\|u - Pf_\delta\| : u \in M_r, \|f_\delta - Au\| \le \delta\}. \tag{1.3}$$

This work is supported by Grant RFBR (project code 10-01-96000_r_ural_a).

Definition 1.1. A set of operator $\{T_\delta : 0 < \delta \le \delta_0\}$ mapping F into U we shall call by *method to find an approximate solution of the equation* (1.1) in the set M_r, if $(\forall \delta \in (0, \delta_0])(T_\delta \in C[F, U])$ and $(T_\delta[f_\delta] \to u_0$ for $\delta \to 0$ uniformly on the M_r such that $\|f_\delta - Au_0\| \le \delta)$.

Definition 1.2. A method $\{T_\delta^{\text{opt}} : 0 < \delta \le \delta_0\}$ we shall call *optimal on the set M_r*, if for each $\delta \in (0, \delta_0]$

$$\Delta(T_\delta^{\text{opt}}) = \Delta_\delta^{\text{opt}}.$$

Immediately [2] we shall define $\omega_1(\tau, r)$ and $\omega(\tau, r)$

$$\omega_1(\tau, r) = \sup\{\|u_1 - u_2\| : u_1, u_2 \in M_r, \|Au_1 - Au_2\| \le \tau\},$$
$$\omega(\tau, M) = \sup\{\|u\| : u \in M_r, \|Au\| \le \tau\},$$

where $\tau \ge 0$.

Using lemma in [3] on the p. 17 we have

$$\omega_1(\tau, r) = \omega(\tau, 2r). \tag{1.4}$$

Lemma 1.3. *If $k \ge 0$ then*

$$\omega(k\tau, kr) = k\omega(\tau, r).$$

Proof. If $k = 0$ then lemma is evident.

If $k > 0$ and $\tau \ge r\|AB\|$ then $k\tau \ge kr\|AB\|$. From definition $\omega(\tau, r)$ it follows that

$$\omega(\tau, r) = r\|AB\|$$

and

$$\omega(k\tau, kr) = kr\|AB\|.$$

Hence $\omega(k\tau, kr) = k\omega(\tau, r)$.

Let $k > 0$ and $\tau < r\|AB\|$, then from $u \in M_r$ $\|Au\| \le \tau$ it follows $ku \in M_{kr}$ and $\|A(ku)\| \le k\tau$. Thus,

$$k\omega(\tau, r) \le \omega(k\tau, kr).$$

In other side, if $u \in M_{kr}$ and $\|Au\| \le k\tau$ then $\frac{u}{k} \in M_r$ and $\|A(\frac{u}{k})\| \le \tau$, the following estimations hold

$$\omega(k\tau, kr) \le k\omega(\tau, r).$$

Two latter inequalities yield the lemma statement. □

Lemma 1.4. *Let* $P \in C[F,U], \omega_1(\tau,r)$ *defined in the* (1.4)*. Then the following estimate holds*

$$\Delta(P) \geq \frac{1}{2}\omega_1(2\delta, r).$$

Proof. Let ε be a sufficiently small positive number. Then from definition $\omega_1(\tau,r)$ it follows that there exist the points $u_1, u_2 \in M_r$ such that

$$\|u_1 - u_2\| \geq \omega_1(2\delta, r) - \varepsilon,$$

and

$$\|Au_1 - Au_2\| \leq 2\delta.$$

If $\overline{f}_\delta = \frac{(Au_1 + Au_2)}{2}$, then we obtain

$$\|Au_1 - \overline{f}_\delta\| \leq \delta, \quad \|Au_2 - \overline{f}_\delta\| \leq \delta.$$

From norm properties we have

$$\max\{\|u_1 - P\overline{f}_\delta\|, \|u_2 - P\overline{f}_\delta\|\} \geq \frac{\|u_1 - u_2\|}{2}.$$

Hence

$$\max\{\|u_1 - P\overline{f}_\delta\|, \|u_2 - P\overline{f}_\delta\|\} \geq \frac{1}{2}(\omega_1(2\delta, r) - \varepsilon), \tag{1.5}$$

from (1.3) it follows that

$$\Delta_\delta(P) \geq \max\{\|u_1 - P\overline{f}_\delta\|, \|u_2 - P\overline{f}_\delta\|\}.$$

Thus, (1.5) and latter inequality yield

$$\Delta_\delta(P) \geq \frac{1}{2}(\omega_1(2\delta, r) - \varepsilon)$$

and, because of arbitrariness of ε, we have the required estimate. □

From (1.2), (1.4), Lemma 1.3 and Lemma 1.4 the following estimate holds

$$\Delta_\delta^{\text{opt}} \geq \omega(\delta, r). \tag{1.6}$$

2 The research of M. M. Lavrent'ev method

Let $U = F = V = H$ be Hilbert spaces. Operators A^*A and BB^* are positive definite. Then [5] for operators A and B we have polar expansions $A = Q\overline{A}$ and $B = \overline{B}P$, where Q and P are unitary operators, $\overline{A} = \sqrt{A^*A}$, $\overline{B} = \sqrt{BB^*}$. Assume that spectrum $Sp(\overline{A})$ of the operator $\overline{A}$ is $[0, \|A\|]$, $\overline{B} = G(\overline{A})$, where $G(\sigma)$ is a continuous strictly increasing function in a $[0, \|A\|]$ and differentiate in a $(0, \|A\|)$, $G(0) = 0$.

Consider equation

$$rG(\sigma)\sigma = \tau; \quad \sigma \in [0, \|A\|]. \tag{2.1}$$

If $0 < \tau < rG(\|A\|)\|A\|$, then (2.1) determines one-to-one implicit functions equation $\bar{\sigma} = \bar{\sigma}(\tau)$, such that $\overline{\sigma}(\tau) \to 0$ when $\tau \to 0$.

Lemma 2.1. *If operators A and B satisfy all conditions above,* $0 < \tau < r\|AB\|$, *then*

$$\omega(\tau, r) = rG[\overline{\sigma}(\tau)],$$

where $\overline{\sigma}(\tau)$ solution of the equation (2.1)

Proof. Let ε be a sufficiently small positive number, $\overline{\sigma}(\tau)$ – solution of the equation (2.1). There exist n_0 such that

$$rG[\overline{\sigma}(\tau)] - rG\left[\frac{n_0 - 1}{n_0}\overline{\sigma}(\tau)\right] < \varepsilon.$$

Consider space H_0

$$H_0 = E_{\overline{\sigma}(\tau)}H - E_{\frac{n_0-1}{n_0}\overline{\sigma}(\tau)}H,$$

where $\{E_\sigma : 0 \le \sigma \le \|A\|\}$ is the composition of the unit generated by operator $\overline{A}$ [4], p. 336.

We note, that

$$M_r = \overline{BS_r}. \tag{2.2}$$

Let $v_0 \in H_0$, $\|v_0\| = r$, then $u_0 = \overline{B}v_0 \in M_r$, $\|u_0\| \ge rG[\overline{\sigma}(\tau)] - \varepsilon$, $\overline{A}u_0 \in H_0$.

As $G(\sigma)$ is strictly increasing function, then

$$\|\overline{A}u_0\| \le rG[\overline{\sigma}(\tau)]\overline{\sigma}(\tau) = \tau.$$

It follows inequality

$$\|u_0\| \leq \omega(\tau, r),$$

and

$$\omega(\tau, r) \geq rG[\overline{\sigma}(\tau)] - \varepsilon.$$

As ε is arbitrary positive number, then

$$\omega(\tau, r) \geq rG[\overline{\sigma}(\tau)]. \tag{2.3}$$

In other side. Let

$$H = H_1 + H_2,$$

where $H_1 = E_{\overline{\sigma}(\tau)}H$ and $H_2 = (E - E_{\overline{\sigma}(\tau)})H$. From theorem, proved in the [4] on p. 336, it follows that subspaces H_1 and H_2 are invariant for the operators $\overline{A}$ and $\overline{B}$.

Let

$$v_0 = v_1 + v_2,$$

where $v_i = pr(v_0, H_i), i = 1, 2$. Then

$$u_i = Bv_i \in H_i; \quad i = 1, 2.$$
$$u_0 = u_1 + u_2,$$
$$\overline{A}u_i \in H_i; \quad i = 1, 2.$$

From here we deduce that

$$\|\overline{A}u_i\| \leq \frac{r_i}{r}\tau; \quad i = 1, 2,$$

where $r_1 = \|v_1\|, r_2 = \|v_2\|$ and $r_1^2 + r_2^2 \leq r^2$.

Overwrite (2.1)

$$r_i G[\overline{\sigma}(\tau)]\overline{\sigma}(\tau) = \frac{r_i}{r}\tau; \quad i = 1, 2.$$

Since $G(\sigma)$ is strictly increasing function, using latter inequalities,

$$\|u_i\| \leq r_i G[\overline{\sigma}(\tau)]; \quad i = 1, 2.$$

From here it follows that

$$\|u_0\| \leq rG[\overline{\sigma}(\tau)].$$

By arbitrariness u_0 from here it follows that

$$\omega(\tau, r) \leq rG[\overline{\sigma}(\tau)]. \tag{2.4}$$

Finally, from (2.3) and (2.4), we obtain

$$\omega(\tau, r) = rG[\overline{\sigma}(\tau)].$$

The proof of lemma is now complete. □

Using results [5], equation (1.1) is equivalent

$$\overline{A}u = g, \quad g = Q^* f. \tag{2.5}$$

Assume that for $g = g_0 \in H$ there exists an exact solution u_0 of the equation (2.5), such that $u_0 \in M_r$. where M_r is defined by formula (2.2). Instead of g_0 such $g_\delta \in F$ is given that $\|g_0 - g_\delta\| \leq \delta$. Number $\delta > 0$ is given.

It is required by M_r, g_δ, δ to find an approximate solution u_δ of equation (2.5) and to evaluate $\|u_0 - u_\delta\|$ on the class M_r.

M. M. Lavrent'ev method [1] uses regularizing set of operators $\{R_\alpha : 0 < \alpha \leq \alpha_0\}$ from H to H:

$$R_\alpha = \overline{B}(\overline{C} + \alpha E)^{-1}, \quad \alpha \in (0, \alpha_0], \tag{2.6}$$

where $\overline{C} = \overline{AB}$.

Lemma 2.2. *For each* $\alpha > 0$ *operator* R_α*, defined in* (2.6)*, is bounded and*

$$\|R_\alpha\| \leq \max_{0 \leq \sigma \leq \|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \alpha}.$$

Proof. R_α is self-adjoint operator.

$$\|R_\alpha\|^2 = \sup_{\|g\| \leq 1} (R_\alpha^2 g, g).$$

Using (2.6) we obtain

$$\|R_\alpha\|^2 = \sup_{\|g\| \leq 1} (\overline{B}^2 [\overline{C} + \alpha E]^{-2} g, g). \tag{2.7}$$

Let $\{E_\sigma : 0 \le \sigma \le \|A\|\}$ – is the composition of the unit generated by operator $\overline{A}$. Then from (2.6) it follows

$$R_\alpha^2 g = \int_0^{\|A\|} \frac{G^2(\sigma)}{[G(\sigma)\sigma + \alpha]^2} dE_\sigma g, \tag{2.8}$$

from (2.7) and (2.8) yield

$$\begin{aligned}\|R_\alpha\|^2 &= \sup_{\|g\|\le 1} \int_0^{\|A\|} \frac{G^2(\sigma)}{[G(\sigma)\sigma + \alpha]^2} d(E_\sigma g, g) \\ &\le \sup_{0\le\sigma\le\|A\|} \frac{G^2(\sigma)}{[G(\sigma)\sigma + \alpha]^2} \sup_{\|g\|\le 1} \int_0^{\|A\|} d(E_\sigma g, g),\end{aligned}$$

i.e.

$$\|R_\alpha\|^2 \le \sup_{0\le\sigma\le\|A\|} \frac{G^2(\sigma)}{[G(\sigma)\sigma + \alpha]^2}. \tag{2.9}$$

As function $\frac{G^2(\sigma)}{[G(\sigma)\sigma+\alpha]^2}$ is continuous in a $[0, \|A\|]$, then exist $\overline{\sigma} \in [0, \|A\|]$ such that

$$\frac{G^2(\overline{\sigma})}{[G(\overline{\sigma})\overline{\sigma} + \alpha]^2} = \sup_{0\le\sigma\le\|A\|} \frac{G^2(\sigma)}{[G(\sigma)\sigma + \alpha]^2}.$$

The latter inequality and (2.9) yields the lemma statement. □

Lemma 2.3. *For each* α *and* $r > 0$ *the following estimate*

$$\sup_{\|v\|\le r} \|R_\alpha \overline{C} v - \overline{B} v\| \le r\alpha \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \alpha}$$

holds.

Proof. As

$$\overline{B}(\overline{C} + \alpha E)^{-1} \overline{C} v - \overline{B} v = -\alpha \overline{B}(\overline{C} + \alpha E)^{-1} v,$$

then using (2.6) it follows that

$$\|R_\alpha \overline{C} v - \overline{B} v\| = \alpha \|\overline{B}(\overline{C} + \alpha E)^{-1} v\|.$$

If $v \ne 0$ then

$$\|R_\alpha \overline{C} v - \overline{B} v\| = \alpha \|v\| \left\| \overline{B}(\overline{C} + \alpha E)^{-1} \frac{v}{\|v\|} \right\|.$$

Hence

$$\sup_{0<\|v\|\le r} \|R_\alpha \overline{C} v - \overline{B} v\| \le r\alpha \sup_{\|w\|\le 1} \|\overline{B}(\overline{C} + \alpha E)^{-1} w\|$$

or

$$\sup_{\|v\|\le r} \|R_\alpha \overline{C} v - \overline{B} v\| \le r\alpha \|R_\alpha\|.$$

The latter inequality and Lemma 2.2 yield the lemma statement. □

Theorem 2.4. *Let for each* $\sigma \in (0, \|A\|)$ $G'(\sigma) > 0$, $\frac{G^2(\sigma)}{G'(\sigma)}$ *is strictly increasing function,* $\delta < rG(\|A\|)\|A\|$, $\overline{\sigma}$ *– solution of the equation* (2.1) *when* $\tau = \delta$, $\overline{\alpha}(\delta) = \frac{G^2(\overline{\sigma})}{G'(\overline{\sigma})}$.

Then

$$\Delta(R_{\overline{\alpha}(\delta)}) \le rG(\overline{\sigma}).$$

Proof. Let $u_0 \in M_r$ and $\|g_\delta - \overline{A} u_0\| \le \delta$.

Then

$$\|u_0 - R_{\overline{\alpha}} g_\delta\| \le \|u_0 - R_{\overline{\alpha}} \overline{A} u_0\| + \delta \|R_{\overline{\alpha}}\| \le \sup_{\|v_0\|\le r} \|R_{\overline{\alpha}} \overline{C} v_0 - B v_0\| + \delta \|R_{\overline{\alpha}}\|.$$

Using the Lemmas 2.2 and 2.3 and latter inequality it is easy to obtain next estimate

$$\|u_0 - R_{\overline{\alpha}(\delta)} g_\delta\| \le (r\overline{\alpha}(\delta) + \delta) \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)}. \tag{2.10}$$

To find maximum of the right part in (2.10), we using the differential calculation methods.

$$\left[\frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)}\right]' = \frac{\overline{\alpha}(\delta) G'(\sigma) - G^2(\sigma)}{[G(\sigma)\sigma + \overline{\alpha}(\delta)]^2}.$$

If $\sigma < \overline{\sigma}$, then

$$\frac{G^2(\overline{\sigma})}{G'(\overline{\sigma})} G'(\sigma) - G^2(\sigma) > 0.$$

If $\sigma = \overline{\sigma}$, then

$$\frac{G^2(\overline{\sigma})}{G'(\overline{\sigma})} G'(\sigma) - G^2(\sigma) = 0.$$

If $\sigma > \overline{\sigma}$, then

$$\frac{G^2(\overline{\sigma})}{G'(\overline{\sigma})} G'(\sigma) - G^2(\sigma) < 0.$$

Hence

$$\max_{0 \le \sigma \le \|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)} = \frac{G(\overline{\sigma})}{G(\overline{\sigma})\overline{\sigma} + \overline{\alpha}(\delta)}.$$

From (2.10) it follows that

$$\|u_0 - R_{\overline{\alpha}(\delta)} g_\delta\| \le r G(\overline{\sigma}).$$

Since arbitrariness u_0 and g_δ, the latter inequality yields the theorem statement. □

Using (1.6), Lemma 2.1 and Theorem 2.4 we obtain

$$\Delta_\delta^{\text{opt}} = r G(\overline{\sigma}),$$

proved next theorem.

Theorem 2.5. *Let for each* $\sigma \in (0, \|A\|)$ $G'(\sigma) > 0$, $\frac{G^2(\sigma)}{G'(\sigma)}$ *is strictly increasing function,* $G(0) = 0$, $0 < \delta \le \delta_0$, $\delta_0 < r G(\|A\|)\|A\|$, $\overline{\sigma}$ *– solution of the equation* (2.1) *when* $\tau = \delta$.

Then M. M. Lavrent'ev method $\{R_{\overline{\alpha}(\delta)} : 0 < \delta \le \delta_0\}$ *with parameter of the regularization*

$$\overline{\alpha}(\delta) = \frac{G^2(\overline{\sigma})}{G'(\overline{\sigma})},$$

defined in the (2.6)*, is optimal.*

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Received December 14, 2009.

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