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The theory of the oscillating cup viscometer filled with two immiscible conductive liquids and placed in the magnetic field

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Abstract. In connection with the experiments investigating the phenomenon of exfoliation in metal melts, a decision of the problem of a torsion viscometer filled with two immiscible conductive liquids and placed in the homogeneous static axial magnetic field is found. The cases when the melt's surface is free and when it touches upon a solid cover are considered.

1. Introduction

There are experiments [1] in which a viscometer was placed into an external magnetic field. However, there was no possibility of quantitative interpretation of such experiments before [2,3]. The generalizing of theoretical basis of the torsional oscillation method for the case when the viscometer is filled with homogeneous liquid and is in a static axial magnetic field is proposed in these works.

In the present work the theory from [2,3] is generalized for the case when the viscometer is filled with two immiscible conductive liquids, which appear, for example, as a result of melt foliation or phase formation on its surface because of impurity segregation, interaction with an atmosphere or with a crucible.

2. Mathematical definition of the problem

Let a vertical cylinder of nonmagnetic dielectric with inner radius R and moment of inertia K be filled with two immiscible conductive liquids and be hanged on an elastic thread. The cylinder is in state of damped torsion oscillations round its axis in uniform static magnetic field with induction $\mathbf{B}_0$ coaxial with the cylinder. We denote kinematic viscosity of the liquids as ν_1 and ν_2, their densities as ρ_1 and ρ_2 and their conductivities as σ_1 and σ_2; let thickness of the bottom layer of liquid be h_1, and that of the top layer be h_2. We suppose also, that the liquids are incompressible and homogeneous. Let also we know oscillation period τ_0 and logarithmic decrement δ_0 of empty cylinder. The problem is finding of oscillation period τ and logarithmic decrement δ of the cylinder filled with the liquids.

Evaluation of magnetic Reynolds number Re_m for liquid metals and typical experimental conditions shows us, that $\mathrm{Re}_m \sim (10^{-4} \div 10^{-5})$. Smallness of Re_m allows us to neglect induced magnetic field as compared to external one, and consider $\mathbf{B} = \mathbf{B}_0$ over the volume of the sample. In this (non-inductive) approximation the equations of magnetic hydrodynamics can be written as:

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$$\Delta\hat{\Phi}_i = \mathbf{B}_0 rot\hat{\mathbf{V}}_i , \tag{1}$$

$$\frac{\partial\hat{\mathbf{V}}_i}{\partial t} + (\hat{\mathbf{V}}_i\nabla)\hat{\mathbf{V}}_i = -\frac{1}{\rho_i}\nabla p_i + \nu_i\Delta\hat{\mathbf{V}}_i + \frac{1}{\rho_i}[\mathbf{j}_i\mathbf{B}_0] , \tag{2}$$

$$\mathbf{j}_i = \sigma_i(-\nabla\hat{\Phi}_i + [\hat{\mathbf{V}}_i,\mathbf{B}_0]) , \tag{3}$$

where Φ is electric potential, $\mathbf{V}$ is the fluid velocity, p is the pressure, and $\mathbf{j}$ is electric current density. Index i=1,2 relates to the number of the fluid. The hat over a variable means that this variable is dimensional, and, therefore, differs from non-dimensional variables which will be introduced later.

While solving the problem we use the following assumptions: 1) oscillation amplitude is sufficiently small to consider only azimuthal liquid velocity component $\hat{V}_{i\varphi}(\hat{r},\hat{z},\hat{t})$; 2) the flow has axial symmetry; 3) we examine the steady state of damped oscillations. To solve the problem we should choose cylindrical coordinate system, the origin of which is placed at the bottom of the cylinder, and z axis is directed along its geometrical axis. Then we introduce non-dimensional values and parameters:

$$t = \hat{t}/\tau , \quad z = \hat{z}/h_1 , \quad r = \hat{r}/R , \quad \gamma = h_1/R , \quad \varepsilon = h_2/h_1 ,$$
$$Re_i = \frac{R^2}{\tau\nu_i}, \quad Ha_i = B_0 R\sqrt{\frac{\sigma_i}{\rho_i\nu_i}}, \quad V_i = \hat{V}_{i\varphi}/\Omega_0 R , \quad \Phi_i = \frac{\hat{\Phi}_i}{B_0 R^2\Omega_0}, \quad \Omega = \hat{\Omega}/\Omega_0 \tag{4}$$

where $\hat{\Omega}$ is the amplitude of the cylinder angular velocity, and Ω_0 is its initial value. Parameters Re and Ha are Reynolds and Hartmann numbers respectively. In the steady state of damped oscillations there should be:

$$V_i = v_i(r,z)e^{-\alpha t} , \qquad \Omega(t) = e^{-\alpha t} , \qquad \Phi_i(r,z,t) = \psi_i(r,z)e^{-\alpha t} , \tag{5}$$

where $\alpha = \delta + 2\pi i$ is non-dimensional complex oscillation frequency. Under assumptions indicated above, using variables (4,5) and taking into account (3) we can rewrite equations (1,2) as:

$$-\alpha\,\mathrm{Re}_i\, v_i = (\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_i}{\partial r}) + \frac{1}{\gamma^2}\frac{\partial^2 v_i}{\partial z^2} - \frac{v_i}{r^2}) + Ha_i^2(\frac{\partial\psi_i}{\partial r} - v_i) \tag{6}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi_i}{\partial r}) + \frac{1}{\gamma^2}\frac{\partial^2\psi_i}{\partial z^2} = (\frac{v_i}{r} + \frac{\partial v_i}{\partial r}) \tag{7}$$

Boundary conditions for values from (6,7) are:

$$v_i(1,z) = 1, \quad v_1(r,0) = r , \quad \left.\frac{\partial v_2}{\partial z}\right|_{z=1+\varepsilon} = 0 , \quad \eta_1\left.\frac{\partial v_1}{\partial z}\right|_{z=1} = \eta_2\left.\frac{\partial v_2}{\partial z}\right|_{z=1} \tag{8-11}$$

$$\left.\frac{\partial\psi_i}{\partial r}\right|_{r=1} = 1, \quad \left.\frac{\partial\psi_1}{\partial z}\right|_{z=0} = 0 , \quad \left.\frac{\partial\psi_2}{\partial z}\right|_{z=1+\varepsilon} = 0 , \quad \sigma_1\left.\frac{\partial\psi_1}{\partial z}\right|_{z=1} = \sigma_2\left.\frac{\partial\psi_2}{\partial z}\right|_{z=1} , \quad \psi_1\big|_{z=1} = \psi_2\big|_{z=1}. \tag{12-16}$$

They mean: (8, 9) is the condition of adhesion of the liquid to solid bounds; (10) is condition of the absence of shearing tensions on the free surface of the upper liquid; (12-14) is the condition of zero-equal normal component of electric current at the outer border of the sample; (15) is the condition of the continuity of normal components of current at the border between liquids; (16) is the condition of the continuity of electric potential at this border. Equations (6, 7) together with the boundary conditions (8-16) definite the boundary problem under consideration.

3. Boundary problem solution

We will search for a solution in the following form

$$v_i = \frac{J_1(k_i r)}{J_1(k_i)} + \chi_i(r,z), \qquad \psi_i = -\frac{1}{k_i}\frac{J_0(k_i r)}{J_1(k_i)} + \xi_i(r,z) , \tag{17}$$

where J_i is i -order Bessel function of the first kind, and the k_i parameter is not yet specified. The boundary conditions (8) and (12) then appear as

$$\chi_i\big|_{r=1} = 0, \qquad \frac{\partial \xi_i}{\partial r}\bigg|_{r=1} = 0, \tag{18}$$

and the equations (6) and (7) are satisfied, if $k_i = \alpha \operatorname{Re}_i$ and

$$\alpha \operatorname{Re}_i \chi_i + \frac{\partial^2 \chi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \chi_i}{\partial r} + \frac{1}{\gamma^2}\frac{\partial^2 \chi_i}{\partial z^2} - \frac{1}{r^2}\chi_i - Ha_i^2(\chi_i - \frac{\partial \xi_i}{\partial r}) = 0 \tag{19}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \xi_i}{\partial r}) + \frac{1}{\gamma^2}\frac{\partial^2 \xi_i}{\partial z^2} = \frac{1}{r}\chi_i + \frac{\partial \chi_i}{\partial r}. \tag{20}$$

Now we represent the functions χ_i и ζ_i as

$$\chi_i = \sum_{n=1}^{\infty} U_{in}(z) J_1(\gamma_n r) \quad , \qquad \xi_i = \sum_{n=1}^{\infty} T_{in}(z) J_0(\gamma_n r), \tag{21}$$

where γ_n is the n-th root of the equation $J_1(x) = 0$. In such salvation form the conditions (18) are met automatically. The rest of the boundary conditions will be satisfied if

$$U_{1n}\big|_{z=0} = -\frac{2\kappa_1^2}{J_0(\gamma_n)\gamma_n(\kappa_1^2 - \gamma_n^2)} = \theta_n, \quad U_{2n} - U_{1n}\big|_{z=1} = \frac{2\gamma_n}{J_0(\gamma_n)}\frac{\kappa_2^2 - \kappa_1^2}{(\kappa_2^2 - \gamma_n^2)(\kappa_1^2 - \gamma_n^2)} = \xi_n, \tag{22,23}$$

$$\frac{dU_{2n}}{dz}\bigg|_{z=1+\varepsilon} = 0, \qquad \eta_1 \frac{dU_{1n}}{dz}\bigg|_{z=1} = \eta_2 \frac{dU_{2n}}{dz}\bigg|_{z=1} \tag{24-25}$$

$$\frac{dT_{1n}}{dz}\bigg|_{z=0} = 0, \quad \frac{dT_{2n}}{dz}\bigg|_{z=1+\varepsilon} = 0, \quad \sigma_1 \frac{dT_{1n}}{dz}\bigg|_{z=1} = \sigma_2 \frac{dT_{2n}}{dz}\bigg|_{z=1}, \tag{26-27}$$

$$T_{2n} - T_{1n}\big|_{z=1} = \frac{2}{J_0(\gamma_n)}\frac{\kappa_1^2 - \kappa_2^2}{(\kappa_1^2 - \gamma_n^2)(\kappa_2^2 - \gamma_n^2)} = -\frac{\xi_n}{\gamma_n}. \tag{28}$$

Substituting (21) into (19) and (20), we receive an equation for $U_{in}(z)$ и $T_{in}(z)$:

$$\frac{\partial^2 U_{in}(z)}{\partial z^2} + \gamma^2(\alpha \operatorname{Re}_i - \gamma_n^2 - Ha_i^2)U_{in}(z) - \gamma^2 \gamma_n Ha_i^2 T_{in}(z) = 0. \tag{29}$$

$$\frac{\partial^2 T_{in}}{\partial z^2} - T_{in}\gamma^2\gamma_n^2 = \gamma^2 \gamma_n U_{in}. \tag{30}$$

Excluding the function $U_{in}(z)$ in (29) with the help of (30), we receive an equation for $T_{in}(z)$:

$$\frac{\partial^4 T_{in}}{\partial z^4} + \kappa_{in}\frac{\partial^2 T_{in}}{\partial z^2} + \Delta_{in} T_{in} = 0, \tag{31}$$

where

$$\kappa_{in} = \gamma^2(\alpha \operatorname{Re}_i - 2\gamma_n^2 - Ha_i^2), \qquad \Delta_{in} = \gamma_n^2 \gamma^4 (\gamma_n^2 - \alpha \operatorname{Re}_i).$$

The solution of the equation (31) is given by

$$T_{1n} = a_{1n} sh[\lambda_{1n}^{+}(1-z)] + b_{1n} ch[\lambda_{1n}^{+}(1-z)] + c_{1n} sh[\lambda_{1n}^{-}(1-z)] + d_{1n} ch[\lambda_{1n}^{-}(1-z)] \tag{25}$$

$$\begin{aligned} T_{2n} &= a_{2n} sh[\lambda_{2n}^{+}(1+\varepsilon-z)] + b_{2n} ch[\lambda_{2n}^{+}(1+\varepsilon-z)] + \\ &+ c_{2n} sh[\lambda_{2n}^{-}(1+\varepsilon-z)] + d_{2n} ch[\lambda_{2n}^{-}(1+\varepsilon-z)] \end{aligned}, \tag{26}$$

where

$$\lambda_{in}^{\pm} = \sqrt{-\frac{\kappa_{in}}{2} \pm \sqrt{\frac{\kappa_{in}^2}{4} - \Delta_{in}}} .$$

If the functions $T_{in}(z)$ are found, then the functions $U_{in}(z)$ can easily be received from (30), and for coefficients a,b,c and d a system of linear algebraic equations is received, with the necessity to comply with the boundary conditions (22-28). The solution of this system is not difficult but bulky, so we do not give the explicit appearance of the coefficients. After they are found the boundary problem solution is completely specified.

4. Viscometric equations

With the knowledge of liquid velocity field, it is easy to count the moment of viscous forces, which influence the cylinder:

$$M = -2\pi\eta_1 R^3 \gamma \Omega_0 e^{-\alpha t} \{M_1 + M_2 + M_3\} , \tag{27}$$

where M_1 and M_2 define the moments applied to the lateral surface of the cylinder from the sides of the liquids 1 and 2 respectively, and M_3 is the moment applied to the bottom:

$$M_1 = \int_0^1 \left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r}\right)\bigg|_{r=1} dz , \quad M_2 = \frac{\eta_2}{\eta_1} \int_1^{1+\varepsilon} \left(\frac{\partial v_2}{\partial r} - \frac{v_2}{r}\right)\bigg|_{r=1} dz , \quad M_3 = -\frac{1}{\gamma^2} \int_0^1 r^2 \left(\frac{\partial v_1}{\partial z}\right)\bigg|_{z=0} dr .$$

After the moment of viscous forces is counted, all other reasoning is similar to those made in the classical work [4]. That is, if the so-called friction function is involved

$$L = \frac{M}{\Omega(t)} = -2\pi\eta_1 R^3 \gamma \{M_1 + M_2 + M_3\} , \tag{28}$$

then the oscillation parameters of the filled crucible appear to be connected to it by the following correlations (viscometric equations):

$$\frac{L'}{K} = p \cdot \left(1 + \frac{p_o^2 + q_o^2}{p^2 + q^2}\right) - 2p_o, \qquad \frac{L''}{K} = q \cdot \left(1 - \frac{p_o^2 + q_o^2}{p^2 + q^2}\right). \tag{29,30}$$

Here L' and L'' are the real and imaginary parts of the function L, $p = \delta / \tau$ is the coefficient of oscillation damping, $q = 2\pi / \tau$ is their cyclic frequency; p_o and q_o are the same as p and q, but they concern the oscillation of the empty crucible.

The correlations (28)–(30) give a full solution of the given problem for the case when the surface of the upper liquid is free. If it borders on a solid cover, the condition (10) should be substituted for the condition (9). Following the reasoning, same to that is given above, we come again to the equations (28, 29) with the only difference that in the right part of the equation (28) an additional component M_4 appear, which describes, as M_3, the friction against the face surface.

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