Краткие сообщения

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ASYMPTOTIC BEHAVIOR OF A DELAY DIFFERENTIAL MODEL IN POPULATION DYNAMICS

L. Berezansky[1], **L. Idels**[2], *brznsky@cs.bgu.ac.il*
[1] *Ben-Gurion University of the Negev, Beer-Sheva, Israel,*
[2] *Vancouver Island University, Nanaimo, Canada*

Considered a scalar nonlinear delay differential equation of the certain species, for which sufficient conditions for oscillation of all solutions and asymptotical stability of the positive equilibrium are obtained.

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1. INTRODUCTION

Consider the following logistic differential equation which is widely used in Population Dynamics

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

Here $N(t)$ is the size of a population, $r \geq 0$ is an intrinsic growth rate, K is a carrying capacity or saturation level. A variety of nonlinear differential equations has been developed to construct numerous models of Mathematical Biology [1–3].

In order to model processes in nature and engineering it is frequently required to know system states in the past. Depending on the phenomena under study the after-effects represent duration of some hidden processes. In general, DDE's exhibit much more complicated dynamics than ODE's since a time lag can change a stable equilibrium into an unstable one and make populations fluctuate, they provide a richer mathematical framework (compared with ordinary differential equations) for the analysis of biosystems dynamics.

Introduction of complex models of Population Dynamics, based on nonlinear DDE's, has received much attention in the literature in recent years.

The application of delay equations to biomodelling is in many cases associated with studies of dynamic phenomena like oscillations, bifurcations, and chaotic behavior. Time delays represent an additional level of complexity that can be incorporated in a more detailed analysis of a particular system.

Delay logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N_\tau}{K}\right). \tag{1}$$

appeared in 1948 in Hutchinson's paper [4]. Here $N_\tau = N(t-\tau)$, $\tau > 0$.

Autonomous equation (1) has been extensively investigated by numerous authors. The first paper on the oscillation of a nonautonomous logistic delay differential equation was published in [5]. Since this publication, the oscillation of the logistic DDE as well as its generalizations were studied by many mathematicians. Some of these results can be found in the monographs [6–8].

It is a well-known fact, that the traditional logistic model in some cases produces artificially complex dynamics, therefore it would be reasonable to get away from the specific logistic form in studying population dynamics and use more general classes of growth models.

For example, in order to drop an unnatural symmetry of the logistic curve, we consider the modified logistic form of Pella and Tomlinson [9], [10] or Richards' growth equation with delay

$$\frac{dN}{dt} = rN\left[1 - \left(\frac{N_\tau}{K}\right)^\gamma\right]. \tag{2}$$

According to [9], $0 < \gamma < 1$ for invertebrate populations (examples of invertebrates are insects, worms, starfish, sponges, squid, plankton, crustaceans, and mollusks), and $\gamma \ge 1$ for the vertebrate populations (these include amphibians, birds, fish, mammals, and reptiles).

In [11] the authors considered Eq. (2) with several delays. They obtained conditions for existence of positive solutions and studied so-called long time average stability. In this paper we obtain oscillation and local stability results for nonautonomous Eq. (2) with several delays.

2. PRELIMINARIES

Our object is a scalar nonlinear delay differential equation

$$\dot N(t) = r(t)N(t)\left[a - \left(\sum_{k=1}^{m} b_k N(g_k(t))\right)^{\gamma}\right],\ t \ge 0, \tag{3}$$

under the following conditions:

(a1) $r(t)$ is a Lebesgue measurable essentially bounded on $[0, \infty)$ function, $r(t) \ge 0$.

(a2) $g_k : [0, \infty) \to R$ are Lebesgue measurable functions, $g_k(t) \le t$, $\lim_{t\to\infty} g_k(t) = \infty$, $k = 1, \ldots, m$.

(a3) $a > 0$, $b_k > 0$, $\gamma > 0$.

Together with (3) we consider for each $t_0 \ge 0$ an initial value problem

$$\dot N(t) = r(t)N(t)\left[a - \left(\sum_{k=1}^{m} b_k N(g_k(t))\right)^{\gamma}\right],\ t \ge t_0, \tag{4}$$

$$N(t) = \varphi(t),\ t < t_0,\ N(t_0) = N_0. \tag{5}$$

We also assume that the following hypothesis holds

(a4) $\varphi : (-\infty, t_0) \to R$ is a Borel measurable bounded function, $\varphi(t) \ge 0$, $N_0 > 0$.

Definition. A locally absolutely continuous function $x : R \to R$ is called *a solution of problem* (4), (5), if it satisfies equation (4) for almost all $t \in [t_0, \infty)$ and equalities (5) for $t \le t_0$.

Lemma 1 [11] *Suppose Conds.(a1)–(a4) hold for equation (3). Then the problem (4), (5) has a unique positive solution N(t), $t \ge t_0$.*

3. OSCILLATION CRITERIA

Definition. We say that a function $y(t)$ is *nonoscillatory* about a number K if $y(t) - K$ is eventually positive or eventually negative. Otherwise $y(t)$ is *oscillatory* about K.

Eq. (3) has a positive equilibrium $N^* = \frac{\alpha^{\frac{1}{\gamma}}}{\sum_{k=1}^{m} b_k}$. In this section we study oscillation of solutions of (3) about N^*.

We will present here some lemmas which will be used in this section.

Consider the linear delay differential equation

$$\dot x(t) + \sum_{k=1}^{l} r_k(t)x(h_k(t)) = 0,\ t \ge 0, \tag{6}$$

and the differential inequalities

$$\dot x(t) + \sum_{k=1}^{l} r_k(t)x(h_k(t)) \le 0,\ t \ge 0, \tag{7}$$

$$\dot x(t) + \sum_{k=1}^{l} r_k(t)x(h_k(t)) \ge 0,\ t \ge 0. \tag{8}$$

Lemma 2 *[6] Let (a1)–(a2) hold for the parameters of Eq. (6). Then the following statements are equivalent:*

1. There exists a non-oscillatory solution of equation (6).

2. There exists an eventually positive solution of the inequality (7).

3. There exists an eventually negative solution of the inequality (8).

Lemma 3 *[6] Let (a1)–(a2) hold for the parameters of Eq. (6). If*

$$\lim_{t\to\infty} inf \int_{max_k h_k(t)}^{t} \sum_{i=1}^{l} r_i(s)ds > 1/e, \tag{9}$$

then all the solutions of equation (6) are oscillatory.

Theorem 1 *Suppose (a1)–(a4) hold and*

$$\int_0^{\infty} r(s)ds = \infty. \tag{10}$$

Then for every nonoscillatory solution N(t) of (3) we have

$$\lim_{t\to\infty} N(t) = N^*. \tag{11}$$

Proof. After a substitution

$$N(t) = N^*(1 + x(t)) \tag{12}$$

Eq. (3) reduced to the following equation

$$\dot{x}(t) = -ar(t)\big(1+x(t)\big)\left[\left(\sum_{k=1}^{m} B_k\left(1+x\big(g_k(t)\big)\right)\right)^{\gamma} - 1\right],\ t \geq 0, \tag{13}$$

where

$$B_k = \frac{b_k}{\sum_{i=1}^{m} b_i}. \tag{14}$$

Condition (a3) implies that $B_k > 0$, $\sum_{k=1}^{m} B_k = 1$.

The zero solution is an equilibrium of Eq. (13), which suits to the equilibrium N^* of Eq. (3).

By Lemma 1 any solution of (3) is positive. Then for any solution of (13) we have $1 + x(t) > 0$. To prove the theorem we have to show that for every nonoscillatory about zero solution of (13) we have

$$\lim_{t\to\infty} x(t) = 0. \tag{15}$$

Suppose $x(t)$ is a nonoscillatory solution of (13). Without loss of generality we can assume that $x(t) > 0$, $t \geq 0$. Hence

$$\left(\sum_{k=1}^{m} B_k(1+x\big(g_k(t)\big))\right)^{\gamma} - 1 \ \geq\ \left(\sum_{k=1}^{m} B_k\right)^{\gamma} - 1 = 0.$$

Then $\dot{x}(t) \ \leq\ 0$ and hence there exists

$$\lim_{t\to\infty} x(t) = l.$$

Suppose $l > 0$. Equality (13) implies

$$x(t) = x(0) - a\int_0^t r(s)(1+x(s))[(\sum_{k=1}^{m} B_k(1+x(g_k(s))))^{\gamma} - 1]ds. \tag{16}$$

If $t \to +\infty$ then the right hand side of (16) tends to $-\infty$, the left hand side has a finite limit. This contradiction proves the theorem.

Theorem 2 *Suppose conditions (a1)–(a4) and (10) hold, $\gamma > 1$ and there exists $\epsilon > 0$ such that all solutions of linear differential equation*

$$\dot{y}(t) = -a\gamma r(t)(1-\epsilon)\sum_{k=1}^{m} B_k y(g_k(t)) \tag{17}$$

are oscillatory, were B_k are denoted by (14).

Then all solutions of (3) are oscillatory about N^*.

Proof. It is sufficient to prove, that all solutions of (13) are oscillatory about zero. Suppose the exists a nonoscillatory solution x of (13). Without loss of generality we can assume, that $x(t) > 0$, $t \geq 0$. Theorem 1 implies, that for some $t_0 > 0$ and for $t \geq t_0$ we have $0 < x(t) < \epsilon$.

Consider the following function

$$f(u_1,\ldots,u_m) = \left(\sum_{k=1}^{m} B_k(1+u_k)\right)^{\gamma} - 1 - \gamma\sum_{k=1}^{m} B_k u_k.$$

We have

$$\frac{df}{du_k} = \gamma\left(\sum_{k=1}^{m} B_k(1+u_k)\right)^{\gamma-1} B_k - \gamma B_k,$$

$$\frac{d^2 f}{du_i du_j} = \gamma(\gamma-1)\left(\sum_{k=1}^{m} B_k(1+u_k)\right)^{\gamma-2} B_i B_j.$$

Hence

$$f(0,\ldots,0) = 0, \frac{df}{du_k}(0,\ldots,0) = 0, \frac{d^2 f}{du_i du_j}(0,\ldots,0) = \gamma(\gamma-1)B_i B_j.$$

Taylor's Formula implies that

$$f(u_1,\ldots,u_m) = \gamma(\gamma-1)\sum_{i=1}^{m}\sum_{j=1}^{m} B_i B_j u_i u_j + o(\Delta u),$$

where

$$\Delta u = \sqrt{\sum_{k=1}^{m} u_k^2}, \lim_{t\to 0}\frac{o(t)}{t} = 0.$$

Then for $u_k \geq 0$, $k = 1, \ldots, m$ and Δu sufficiently small $f(u_1, \ldots, u_m) \geq 0$. Hence for ϵ small enough we have

$$\dot{x}(t) \leq -a\gamma r(t)(1-\epsilon)\sum_{k=1}^{m} B_k x\big(g_k(s)\big),\ t \geq 0.$$

Lemma 2 implies now that Eq. (17) has a nonoscillatory solution. We have a contradiction with our assumption. The theorem is proven.

Corollary 2.1 *Suppose conditions (a1)–(a4) and (10) hold, $\gamma > 1$,*

$$\lim_{t\to\infty}\inf a\gamma\int_{max_k g_k(t)}^{t} r(s)ds > 1/e. \tag{18}$$

Then all solutions of (3) are oscillatory about N^*.

Proof Inequality (18) implies, that for some $\epsilon > 0$

$$\lim_{t\to\infty}\inf a\gamma\,(1-\epsilon)\int_{max_k g_k(t)}^{t}\sum_{i=1}^{m} B_i\, r(s)ds > 1/e.$$

Lemma 3 and Theorem 2 imply this corollary.

4. ASYMPTOTIC STABILITY

Consider a general nonlinear delay differential equation

$$\dot{x}(t) = f(t, x(t), x(g_1(t)), \dots, x(g_m(t))),\ t \ge 0, \tag{19}$$

with the initial function and the initial value

$$x(t) = \varphi(t),\ t < 0,\ x(0) = x_0, \tag{20}$$

under the following conditions:

(b1) $f(t, u_0, u_1, \dots, u_m)$ satisfies Caratheodory conditions: it is Lebesgue measurable in the first argument and continuous in other arguments, $f(t, 0, \dots, 0) = K$;

(b2) $g_k(t)$ are Lebesgue measurable functions,

$g_k(t) \le t, \sup_{t\ge 0}[t - g_k(t)] < \infty$;

(b3) $\varphi : (-\infty, 0) \to R$ is a Borel measurable bounded function.

We will assume that the initial value problem (19)–(20) has a unique global solution $x(t)$, $t \ge 0$.

Definition. We will say that the equilibrium K of Eq. (19) is *(locally) stable*, if for any $\epsilon > 0$ there exists $\sigma > 0$ such that for every initial conditions $|x(0)| < \sigma_0$, $|\varphi(t)| < \sigma_0$, $\sigma_0 \le \sigma$, for the solution $x(t)$ of (19)–(20) we have $|x(t) - K| < \epsilon$, $t \ge 0$.

If, in addition, $\lim_{t\to\infty}(x(t) - K) = 0$, then the equilibrium K of Eq. (19) is *(locally) asymptotically stable*.

Suppose there exist $M > 0$, $\gamma > 0$ such that

$$|x(t) - K| \le Mexp\{-\gamma t\}(|x(0)| + \sup_{t<0}|\varphi(t)|)$$

for all $x(0)$ and $\varphi(t)$ such that $|x(0)| + \sup_{t<0}|\varphi(t)|$ is sufficiently small. Then we will say that the equilibrium K of Eq. (19) is exponentially stable.

Lemma 4 *[12] Suppose (a1), (b2), (b3) hold for linear equation (6) and*

$$\lim_{t\to\infty} sup \sum_{k=1}^{l} r_k(t)\big(t - h_k(t)\big) < 1.$$

Then Eq. (6) is exponentially stable.

Lemma 5 *[13], [14] Suppose that (b1)–(b3) hold, for sufficiently small u if $|u_k| \le u$, $k = 0, \dots, m$ then*

$$\left|f(t, u_0, \dots, u_m) - \sum_{k=0}^{m} \frac{dF}{du_k}(t, K, \dots, K)u_k\right| = o(u),$$

where $\lim_{u\to 0}\frac{o(u)}{u} = 0$.

If the linear equation

$$\dot{y}(t) = \sum_{k=0}^{m} \frac{dF}{du_k}(t, 0, \dots, 0)y(g_k(t))$$

is exponentially stable, then the equilibrium K of Eq. (19) is locally asymptotically stable.

Theorem 3 *Suppose that for equation (3) Conds. (a1), (a3), (b2), (b3) hold and*

$$\lim_{t\to\infty} \sup a\gamma r(t) \sum_{k=1}^{m} B_k\big(t - g_k(t)\big) < 1. \tag{21}$$

Then equilibrium N^* of Eq.(3) is asymptotically stable.

Proof. A substitution $N(t) = N^*(1 + x(t))$ implies that equilibrium N^* of Eq. (3) is asymptotically stable if and only if the zero solution of (13) is asymptotically stable.

Lemma 4 and inequality (21) imply that linear equation

$$\dot{x}(t) = -a\gamma r(t) \sum_{k=1}^{m} B_k x\big(g_k(t)\big)$$

is exponentially stable. Lemma 5 implies now that the zero solution of (13) is asymptotically stable.

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АСИМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ ДИФФЕРЕНЦИАЛЬНОЙ МОДЕЛИ С ЗАДЕРЖКОЙ В ДЕМОГРАФИЧЕСКОЙ ДИНАМИКЕ

Л. Березанский[1], Л. Айделс[2]
[1] Университет имени Бен-Гуриона, Беэр-Шева, Израиль,
[2] Университет острова Ванкувер, Нанаймо, Канада

Рассмотрено скалярное нелинейное дифференциальное уравнение определенного типа с задержкой, для которого получены достаточные условия колебания всех решений и асимптотической стабильности положительного равновесия.

Ключевые слова: дифференциальные уравнения с задержкой, нелинейность Ричарда, колебание, стабильность.

Березанский Леонид, отделение математики, Университет имени Бен-Гуриона, Беэр-Шева, Израиль; brznsky@cs.bgu.ac.il.

Айделс Лев, отделение математики, Университет острова Ванкувер, Нанаймо, Канада.

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