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INVESTIGATION OF THE UNSTEADY-STATE HYDRAULIC NETWORKS BY MEANS OF SINGULAR SYSTEMS OF INTEGRAL DIFFERENTIAL EQUATIONS

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Analysis of complex hydraulic networks, electric circuits, electronic schemes, chemical processes etc. often results in a system of interconnected differential and algebraic equations. If the process under study has after-effect, then the system includes integral equations. This paper addresses simulation of hydraulic networks by means of the theory for singular systems of integral differential equations. We present theoretical tools that help investigate qualitative properties of such systems and search for effective methods of solution. A mathematical model for the straight through boiler circuit has been developed and a numerical method for its solution has been constructed. Experimental results showed that the theory for singular systems of integral differential equations performs well when applied to simulation of the hydraulic networks.

Keywords: differential-algebraic equations; integral-algebraic equations; hydraulic network; hydraulic circuit; index; numerical methods.

Introduction

Consider systems of ordinary differential equations

$$A\dot{x} + F(x,t,\nu) = 0,\ t \in I_t = (-\infty, +\infty), \tag{1}$$

where A is a $(\varrho \times \varrho)$-matrix with constant components, $F(x,t,\nu)$ is an n-dimensional vector-function, $x \equiv x(t)$ is a desired vector-function, $\nu \in \aleph = (-\nu_0, \nu_0)$ is a numeric variable. For the sake of simplicity, it is assumed that $F(x,t,\nu) \in \mathbf{C}^\infty(\mathbf{R}^\varrho \times I_t \times \aleph)$ and

$$\det A = 0. \tag{2}$$

Systems (1) satisfying (2) are commonly called differential algebraic equations (DAEs) [1], other frequently used terms include singular systems [2] and algebraic differential systems [3]. Such systems appear in many applications, for example, in electronic schemes, electric and hydraulic networks, mechanical systems etc. [1–5]. In the works [6–8], systems (1), describing various substantial problems (A is a non null-kernel operator), have been studied in infinite-dimensional spaces.

By the solution of system (1) for a fixed value of $\nu = \nu_*$ on $T = (\alpha, \beta) \subseteq I_t$ we mean the vector-function $x \equiv x(t) \in \mathbf{C}^1(T)$ which reduces (1) into identity on T.

Here and in what follows, we use the uniform norm of the m-dimensional vector $b \in \mathbf{R}^\varrho$ and the corresponding norm of the $(\varrho \times \varrho)$-matrix $B = \left\{b_{ij}, i, j = \overline{1,\varrho}\right\}$ which are found by the rules $\|b\| = \max\left\{|b_i|, i = 1, 2, ..., \varrho\right\},\ b = (b_1, b_2, ..., b_\varrho)^\top$, $\|B\| = \max\{\sum_{j=1}^{\varrho} |b_{ij}|, i = 1, 2, \dots, \varrho\}$, where $\top$ stands for conjugation.

It is important to mention that dependence on t sometimes might be omitted if it does not lead to misunderstanding.

1. Mathematical Models for Hydraulic Networks

A hydraulic network is a system comprising an interconnected set of discrete components that transport media (such as gas, liquid or a mixture of gas and liquid). The network usually comprises the following components: active components (hydraulic power pack, e.g. pumps), transmission lines (e.g. pipes) and passive components (hydraulic cylinders). Its scheme can be presented in the form of a finite oriented graph with explanatory statements if required. The number of nodes n and the number of lines m are said to be the parameters of the hydraulic network under study.

The hydraulic network graph can be presented by a full $(m \times n)$-matrix $\mathcal{A}$ of nodes and lines that identically describes the structure and the orientation of the network: $a_{ji} = 1$, if the line i comes from the node j; $a_{ji} = -1$, if the line i comes into node j; $a_{ji} = 0$ if the node j does not belong to the line i. The finite closed set of the oriented lines, where only the start and the end nodes coincide, is called a simple circuit. We say that the line is active if it connects nodes of active components, otherwise we call this line passive [9].

The matrix of nodes and lines should be considered along with the $(l \times n)$-matrix $\mathcal{B}$ of the network circuits which describes the basic system of circuits and lines: $b_{ji} = 1$, if the line i belongs to the circuit and its orientation coincide with the direction of the circuit, otherwise $b_{ji} = -1$; $b_{ji} = 0$ if the line i does not belong to the circuit.

Introduce the following vector-functions: $X(t) = \Big(x_1(t)\ x_2(t) \dots\ x_n(t)\Big)^\top$ is the flow rate in pipelines; $P(t) = \Big(p_1(t)\ p_2(t) \dots\ p_l(t)\Big)^\top$ denotes pressures at nodes; $y_i(t) = p_i(t) - p_{i+1}(t)$ is a pressure drop in the i-th pipeline; $P^*(t) = \Big(p^*_{l+1}(t)\ p^*_{l+2}(t) \dots\ p^*_m(t)\Big)^\top$ denotes the known pressures; $H(t) = \Big(\tilde{h}_1(t)\ \tilde{h}_2(t) \dots\ \tilde{h}_n(t)\Big)^\top$ is the hydraulic heads in pipelines; $Q(t) = \Big(q_1(t)\ q_2(t) \dots\ q_m(t)\Big)^\top$ is the inflows (outflows) at nodes; $h_i(t) = y_i(t) + \tilde{h}_i(t)$ is a pressure loss in the i-th line due to friction, $y_i(t) = h_i(t)$ for a passive line.

Here we establish two Kirchhoff's laws:

$$\mathcal{A}X(t) = Q(t),\ \ \mathcal{B}Y(t) = 0, \tag{3}$$

where $Y(t)$ is a vector-function of the pressure drops in pipelines. The first law reflects the fact that for any node j the inflow equals to the outflow: $\sum\limits_j x_i(t) = q_i(t),\ \sum\limits_{i=1}^m q_i(t) = 0$. The second law implies that in any closed circuit q the total pressure drop equals to zero: $\sum\limits_q y_i(t) = 0$. In system (3), the number of variables is greater than the number of equations. To amend this, it is common to use so called closing relations which describe the movement of the media along the pipelines:

$$y_i(t) + \tilde{h}_i(t) = s_{0i}x_i(t) + s_{1i}x_i^2(t), \quad s_{0i} > 0, \quad s_{1i} > 0, \tag{4}$$

where $s_{0i},\ s_{1i}$ are the pipe frictions corresponding to the stream-line and turbulent flows [9]. After some obvious transformations, we arrive at the closed system of nonlinear equations

$$\begin{pmatrix} S_0 & -\mathcal{A}_1^\top \\ \mathcal{A}_1 & 0 \end{pmatrix} \begin{pmatrix} X(t) \\ P(t) \end{pmatrix} + \begin{pmatrix} S_1 \left|X(t)\right| X(t) \\ 0 \end{pmatrix} = \begin{pmatrix} H(t) + \mathcal{A}_2^\top P^*(t) \\ Q_1(t) \end{pmatrix}, \tag{5}$$

where $|X(t)|\,X(t) = \mathrm{diag}\{x_1(t)|x_1(t)|, x_2(t)|x_2(t)|, \ldots, x_n(t)|x_n(t)|\}$, $Q_1(t)$ is a vector-function of inflows at the nodes with unknown pressures, $S_0 = \mathrm{diag}\{s_{01}, s_{02}, \ldots, s_{0n}\}$ and $S_1 = \mathrm{diag}\{s_{11}, s_{12}, \ldots, s_{1n}\}$, $(\mathcal{A}_1^\top\ \ \mathcal{A}_2^\top) = \mathcal{A}^\top$, the matrix $\mathcal{A}_1$ is full rank. If we find $H(t)$, $P^*(t)$, $Q_1(t)$ at a given time t and solve (5) by Newton's method, we obtain distribution for the flows $X(t)$ and the pressures $P(t)$.

Notice that the pipe frictions may depend on $X(t)$, $P(t)$ [10, 11]. In the works by A.P. Merenkov, it was proposed to replace $x_i^2(t)$ by $|x_i(t)|x_i(t)$ in (4). This enables us to carry computations when $y_i(t)+\tilde{h}_i(t)$ alternates in signs. For example, set $s_{0i}=0$, $[y_i(t)+\tilde{h}_i(t)]<0$. Then (4) does not have a real solution with respect to x_i, whereupon system $y_i(t)+\tilde{h}_i(t) = s_{1i}|x_i(t)|x_i(t)$ does.

In the monograph [5], the following closing relations were obtained by integrating over the space of the general motion equations: $y_i(t)+\tilde{h}_i(t) = r_i(t)\dot{x}_i(t) + s_{0i}x_i(t) + s_{1i}x_i^2(t)$, $r_i(t) > \bar{\kappa} = \mathit{const} > 0$, $t\in I_t$, $i=\overline{1,n}$. In accordance with the technique developed by A.P. Merenkov, we replace this with

$$y_i(t)+\tilde{h}_i(t) = r_i(t)\dot{x}_i(t) + s_{0i}x_i(t) + s_{1i}|x_i(t)|x_i(t), \tag{6}$$

which corresponds to the DAE

$$\begin{pmatrix} R(t) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\dot{X}(t)\\ \dot{P}(t)\end{pmatrix} + \begin{pmatrix} S_0 & -\mathcal{A}_1^\top\\ \mathcal{A}_1 & 0\end{pmatrix}\begin{pmatrix} X(t)\\ P(t)\end{pmatrix} + \begin{pmatrix} S_1\,|X(t)|\,X(t)\\ 0\end{pmatrix} =$$

$$= \begin{pmatrix} H(t)+\mathcal{A}_2^\top P^*(t)\\ Q_1(t)\end{pmatrix}, \tag{7}$$

where $R(t) = \mathrm{diag}\{r_1(t), r_2(t), \ldots, r_n(t)\}$ are the momentum parameters that depend on the geometric features of the given part of the circuit. Some pipelines may have regulator components and, therefore, taking into account (6), the line equation has the form

$$y_i(t)+h_i(t) = r_i(t)\dot{x}_i(t) + s_{0i}x_i(t) + [s_{1i} + \kappa_i\int_0^t(\psi_i(x_i(\tau))-\theta_i)d\tau]x_i(t)\,|x_i(t)|\,, \tag{8}$$

where κ_i is a proportionality factor, θ_i is a value for the regulator component, $\psi_i(.)$ is the regulator function.

Hence, taking into account (8), system (7) with automatic regulator components can be presented in the form of a singular vector-valued integral differential equation

$$W(y) := \mathcal{R}(t)\dot{y} + \Psi(y, \mathcal{V}y, t) = 0,\ t\in T=[\alpha,\beta]\subset[0,\ \infty), \tag{9}$$

where $y = (X^\top\ \ P^\top)^\top$, $\mathcal{V}y = \int_0^t K(t,\tau,y(\tau))d\tau$ is the Volterra operator, $\mathcal{R}(t) = \mathrm{diag}(R(t), 0)$.

Example 1. Consider the straight-through boiler circuit (see Fig. 1). Along the circuit, the water is pumped at node 5, then it is heated and turns into steam. Afterwards, the steam is heated to 545^oC and passes to the turbine through the valve. It is shown on Fig. 1 that the circuit has 6 segments: $x_1, x_2, x_3, x_4, x_5, x_6$. Water flows through the segments x_1, x_3 which also have regulators for the feed water and the live steam temperature. The

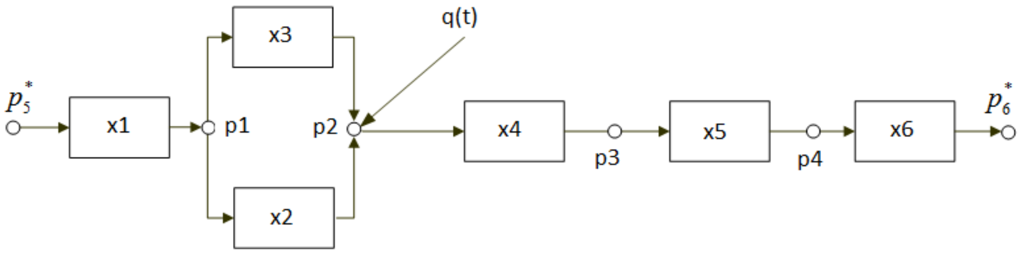

Fig 1. Diagram for the straight-through boiler circuit

segment x_2 has the water-steam mixture coming through it, whereas steam passes through x_4 and x_6. The control valve of the turbine is located at x_5. The inflow $q(t)$ simulates the media density variation at the boiling segment.

Write down the flow rate equations using the first Kirchhoff's law for the nodes p_1, p_2, p_3, p_4 (Fig. 1):

$$x_1 - x_2 - x_3 = 0; \qquad x_2 + x_3 - x_4 = q; \;\; x_4 - x_5 = 0; \qquad x_5 - x_6 = 0. \tag{10}$$

The pressure loss equations have the form

$$p_5^* - p_1 = r_1\dot{x}_1 + [s_{11} + \kappa_1 \int_0^t (x_1(\tau) - \theta_1)d\tau]\,|x_1|\,x_1;\;\; \kappa_1 = 0,003,\;\; p_1 - p_2 = r_2\dot{x}_2 + s_{12}\,|x_2|\,x_2;$$

$$p_1 - p_2 = r_3\dot{x}_3 + [s_{13} + \kappa_3 \int_0^t (x_3(\tau) - \theta_3)d\tau]\,|x_3|\,x_3;$$

$$\kappa_3 = 0,001,\;\; p_2 - p_3 = r_3\dot{x}_3 + s_{14}\,|x_4|\,x_4/(p_2 + p_3);$$

$$p_3 - p_4 = r_5\dot{x}_5 + s_{15}\,|x_5|\,x_5/(p_4 - 0,09p_3);\;\; p_4 - p_6^* = r_6\dot{x}_6 + s_{16}\,|x_6|\,x_6/(p_6^* + p_4). \tag{11}$$

In (7) we have:

$$\mathcal{A}_1^\top = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \mathcal{A}_2^\top = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Values for the regulators are set: $\theta_1 = 75$, $\theta_3 = 9$. The flow rate is measured in kilograms per second, pressure is measured in units of atmosphere, time is measured in seconds.

2. Basic Definitions

Before we start to analyse the system obtained, let us introduce the essential for the future investigation notation.

Definition 1. *Let*

$$\Lambda_1(x) := A\dot{x} + \Phi(x,t) = 0,\ \Phi(x,t) = F(x,t,\nu_*),\ t \in I_t = (-\infty, +\infty), \qquad (12)$$

where $\Phi(x,t) \in \mathbf{C}^{(2,2)}_{tx}(\mathbf{R}^\varrho \times I_t)$, $\nu_* \in N = (-\nu_0, \nu_0)$ *is a fixed value of the parameter, and, as in [12], the following conditions hold:*
1. All solutions $x(t;t_0;x_0)$, *where* $x_0 \in \mathcal{M}$, $\mathcal{M} \subseteq \mathbf{R}^\varrho$ *is some mapping, are defined when* $t_0 \le t < \infty$;
2. There exists a unique solution $\eta(t)$, $t \in I_t$ *bounded along the entire real axis:* $\sup\{\|\eta(t)\|, t \in I_t\} = \mu < \infty$;
3. For each solution $x(t;t_0;x_0)$, $\lim_{t\to\infty}[x(t;t_0;x_0) - \eta(t)] = 0$.
Then, system (12) possesses the convergence property.

By combining definitions from [2,3,13], introduce the following notion and a statement.

Definition 2. *1. The sum* $\lambda A + B$, *where* A, B *are matrices of the equal dimensions,* λ *is a scalar (generally, complex) parameter, is called a matrix pencil;*
2. The pencil of $(\varrho\times\varrho)$*-matrices* $\lambda A + B$ *is regular if there exists a value for the parameter* λ_0 *such that* $\det(\lambda_0 A + B) \neq 0$;
3. The smallest possible integer positive number k, *starting with which*

$$\mathrm{rank} W^k = \mathrm{rank} W^{k+1},\ W = \left[(\lambda_0 A + B)^{-1} A\right], \qquad (13)$$

is called an index of the matrix pencil $\lambda A + B$;
4. The regular matrix pencil $\lambda A + B$ *is said to satisfy the rank-degree criterion if*

$$\deg\ \det[\lambda A + B] = \mathrm{rank} A = r.$$

Lemma 1. *If the pencil of square matrices* $\lambda A + B$ *is regular, then there exist such matrices* P *and* Q *with constant components that*

$$P(\lambda A + B)Q = \lambda \begin{pmatrix} E_d & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & E_{\varrho-d} \end{pmatrix},$$

where N *is a nilpotent matrix (i.e. for some* $j \ge k$, $N^j = 0$*).*

DAEs possess a complex inner structure. To measure this complexity to some extent, it is common to use the notation of index. Various definitions of index can be found in literature (see, e.g., [1], [3], [14]), however, below we use the definition from [14].

Definition 3. *Let there exists a differential operator* $\Lambda_l(z) := \sum_{j=0}^{l} L_j(t,z,\dots,z^{(l-2)})(d/dt)^j$, *where* $L_j \in \mathbf{C}(T \times \mathbf{R}^{\varrho(l-2)})$ *are* $(\varrho\times\varrho)$*-matrices with the property* $\Lambda_l(z) \circ \Lambda_1(y) = \tilde{A}(z,t)\dot{z} + \tilde{\Phi}(z,t)\ \forall z \equiv z(t) \in \mathbf{C}^{l+1}(T)$, *where* $\det \tilde{A}(\upsilon,t) \neq 0\ \forall(\upsilon,t) \in \mathbf{R}^\varrho \times T$. *The smallest possible* l *is said to be the index of system (12) on* T. *When* $l - 2 < 0$, *matrices* L_j *do not depend on* t *and* z.

For the time-invariant system (12)

$$A\dot{x}(t) + Bx(t) = f(t), \qquad (14)$$

where the pencil of square matrices $\lambda A + B$ is regular and $f(t)$ is a known vector-function, we can assume, by means of Lemma 1, $\Lambda_l = Q\mathrm{diag}\{E_d, \sum\limits_{j=0}^{l}(-N)^j(d/dt)^{j+1}\}P$. In other words, $l = k$, i.e. the index equals to the parameter from (13).

Definition 4. *Let there exist an integral differential operator*

$$\tilde{\Lambda}_l(z) := \sum_{j=0}^{l} L_j(t, z, \dots, z^{(l-2)}, \mathcal{V}, \mathcal{V}_1, ..., \mathcal{V}_{l-2})(d/dt)^j,$$

where $L_j \in \mathbf{C}(T \times \mathbf{R}^{\varrho(2l-2)})$ *are* $(\varrho \times \varrho)$*-matrices,* $\mathcal{V}_j$ *are the Volterra operators with the property* $\tilde{\Lambda}_l(z) \circ W(z) = \tilde{\mathcal{R}}(z, t, \mathcal{V}, \mathcal{V}_1, ..., \mathcal{V}_{l-2})\dot{z} + \tilde{\Psi}(z, t, \mathcal{V}, \mathcal{V}_1, ..., \mathcal{V}_{l-1})$ $\forall z \equiv z(t) \in \mathbf{C}^{l+1}(T)$*, where* $\det \tilde{\mathcal{R}}(v, t, w, w_1, ..., w_{l-2}) \neq 0$ $\forall (v, t, w, w_1, ..., w_{l-2}) \in \mathbf{R}^{n(2l-2)} \times T)$*. The smallest possible* l *is called the index of system (9) on* T*. If* $l - 2 < 0$*, the matrices* L_j *depend only on* t*. In some cases it is safe to assume* $\mathcal{V}_j z = \int_0^t [\partial^j K(t, \tau, y(\tau))/\partial t^j] d\tau$*.*

3. Linear DAEs with the Convergence Property

Lemma 2. *If system (12) possesses the property of convergence and* $\Phi(x, t) \equiv \Phi(x, t+\omega)$*, then the bounded solution* $\eta(t)$ *is also* ω*-periodic with respect to* t*.*

Proof. Indeed, let $\Phi(x, t) \equiv \Phi(x, t + \omega)$. We have $A(d/dt)\eta(t + \omega) = A\dot{\eta}(t + \omega) = \Phi(\eta(t + \omega), t + \omega) = \Phi(\eta(t + \omega), t)$. Therefore, $\eta(t + \omega)$ is a bounded solution on I_t. By definition, $\eta(t)$ is unique. Hence, $\eta(t + \omega) = \eta(t)$.

□

Theorem 1. *Let system (14) satisfy the conditions:*
1. The matrix pencil $\lambda A + B$ *is regular;*
2. $f(t) \in \mathbf{C}^k(I_t)$*,* $\|f^{(j)}(t)\| \leq \kappa$*,* $\kappa = const > 0$*,* $j = \overline{0, k-1}$*,* $f^{(0)}(t) = f(t)$*, where* k *is the index of* $\lambda A + B$*;*
3. All roots of the polynomial $\det(\lambda A + B)$ *have negative real parts.*
Then, system (14) possesses the convergence property and

$$\eta(t) = \sum_{j=0}^{k-1} S_j f^{(j)}(t) + \int_{-\infty}^{t} G_1(t - \tau) f(\tau) d\tau. \tag{15}$$

Here $G_1(v)$*,* S_j *are some* $(\varrho \times \varrho)$*-matrices defined below.*

Proof. Multiply system (14) on the left by P and introduce the change of variables $x = Qy$, where P and Q are the matrices from Lemma 1. We obtain

$$\begin{pmatrix} E_d & 0 \\ 0 & N \end{pmatrix} \dot{y} + \begin{pmatrix} J & 0 \\ 0 & E_{\varrho-d} \end{pmatrix} y = \tilde{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \tilde{f}(t) = Pf(t). \tag{16}$$

Introduce the splitting $y = (y_1^\top, y_2^\top)^\top$. Consider the first subsystem of (16) $\dot{y}_1 + Jy_1 = f_1(t)$. In accordance with [12], under condition 3 of the theorem all real numbers of the matrix J are negative and the unique bounded solution of the first subsystem is defined by the formula

$$y_1(t) = \int_{-\infty}^{t} e^{-J(t-\tau)} f_1(\tau) d\tau.$$

For the second subsystem $N\dot{y}_2 + y_2 = f_2(t)$, a simple substitution verifies that

$$y_2(t) = f_2(t) + \sum_{j=1}^{k-1} (-N)^j f_2^{(j)}(t).$$

Then, $G_1(v) = Q\text{diag}\left\{e^{-Jv}, 0\right\} P; S_j = Q\text{diag}\left\{0, (-N)^j\right\}$.

□

Similarly, we can prove a more general statement using (16).

Theorem 2. *Let system (14) satisfy conditions 1 and 2 of Lemma 3 and all roots of the polynomial* $\det(\lambda A + B)$ *have non-zero real parts.*
Then, there exists some matrix $G(t) \in \mathbf{C}^\infty(0 < |t| < \infty)$ *with the properties:*
1. $G(+0) - G(-0) = M$, *where* M *is a* $(\varrho \times \varrho)$*-matrix defined below;*
2. $\|G(t)\| \le ce^{-\alpha|t|}, (t \ne 0)$, *where* c *and* α *are positive constants;*
3. $A\dot{G}(t) + BG(t) = 0$, $t \ne 0$;
4. The expression

$$\eta(t) = \sum_{j=0}^{k-1} S_j f^j(t) + \int_{-\infty}^{+\infty} G(t-\tau) f(\tau) d\tau \tag{17}$$

is a unique bounded solution of (14) on I_t.

Lemma 3. *For the bounded solution* $\eta(t)$ *of system (14) under the conditions of Theorems 1 and 2, the following estimation holds*

$$\sup_t \|\eta(t)\| \le \tilde{\kappa} \sup_t \sum_{j=0}^{k-1} \left\| f^{(j)}(t) \right\|, \ \tilde{\kappa} = const \ge 0.$$

If the free term $f(t)$ *of (14) is an* ω*-periodic function* $f(t+\omega) \equiv f(t) (\omega > 0)$, *then the bounded solution* $\eta(t)$ *is also* ω*-periodic.*

4. Quasilinear DAEs with the Convergence Property

Definition 5. *(see, e.g., [2]) The* $(\varrho_1 \times \varrho)$*-matrix* A^- *is said to be semi-inverse for the* $(\varrho \times \varrho_1)$*-matrix* A *if* $AA^-A = A$.

The semi-inverse matrix is defined for any $(\varrho \times \varrho_1)$-matrix A and techniques for its computation are well developed [2].

Lemma 4. *[17] Let:*
1. In system (1) $F(x, t, \nu) = F(x)$;
2. $F(x) \in \mathbf{C}^2(U)$, $U = \{x : \|x - a\| \le \rho\}$, *where* $a \in \mathbf{R}^\varrho$ *and* $F(a) = 0$;
3. The matrix pencil $\lambda A + F_x(a)$, *where* $F_x = \partial F(x)/\partial x$, *satisfies the rank-degree criterion.*
Then, the first r *roots of the polynomial* $\det\left[\lambda E_\varrho - \partial F(a)/\partial x\right]$ *are equal to the roots of* $\det\left[\lambda A + F_x(a)\right]$, *and the rest* $\varrho - r$ *roots are equal to -1, where*

$$F(x) = -\left[A + SF_x(x)\right]^{-1} F(x), S = E_\varrho - AA^-.$$

Lemma 5. *Let in system (12)* $\Phi(x,t) \in \mathbf{C}^{(2,2)}_{tx}(\mathbf{R}^{\varrho} \times I_t)$. *Then, system (12) has index 1 if* $\det[\lambda A + \Phi_x(x,t)] = a_r(x,t)\lambda^r + \ldots$, *where* $r = \mathrm{rank} A$, $\Phi_x = \partial\Phi/(x,t)\partial x$, *and* $a_r(x,t) \neq 0\ \forall (x,t) \in \mathbf{R}^{\varrho} \times I_t$.

Proof. Consider the equality

$$P\left[A + SP^{-1}P\Phi_x(Qy,t)\right]Q\dot{y} + P\Phi(Qy,t) + P\Phi_t(Qy,t) = 0,$$

where $\Phi_t = \partial\Phi/\partial t, x = Qy, PAQ = \mathrm{diag}\{E_r, 0\}, \det(PQ) \neq 0$. It is clear that

$$\det\begin{pmatrix} E_r & 0 \\ B_{21}(y,t) & B_{21}(y,t) \end{pmatrix} = \det P\left[A + SP^{-1}P\Phi_x(Qy,t)\right]Q,$$

where $\begin{pmatrix} B_{11}(y,t) & B_{12}(y,t) \\ B_{21}(y,t) & B_{22}(y,t) \end{pmatrix} = P\Phi_x(Qy,t)Q$. Since the matrix $S = E_{\varrho} - AA^{-}$ is a projector: $S^2 = S$, then, by the definition for a semi-inverse matrix, we can choose such P that $PSQ = \mathrm{diag}\{0, E_{\varrho - r}\}$. It follows that $a_r(x,t) = \det B_{22}(Q^{-1}x,t)/\det(PQ)$. Therefore, by applying $E_{\varrho} + S(d/dt)$ to (12) and multiplying by $[A + S\Phi_x(x,t)]^{-1}$, we obtain a system in the normal form

$$\dot{x} = F(x,t) = -\left[A + S\Phi_x(x,t)\right]^{-1}\left[\Phi(x,t) + \Phi_t(x,t)\right]. \tag{18}$$

□

Theorem 3. *Let system (18) satisfy the conditions:*
1. $\sup\{F(0,t),\ t \in I_t\} = \kappa < \infty$;
2. The biggest eigenvalue $\lambda_{\max}(x,t)$ *of the matrix* $\left[F_x(x,t) + F_x^{\top}(x,t)\right]/2, F_x(x,t) = \partial F/(x,t)\partial x$ *is such that* $\lambda_{\max}(x,t) \leq \epsilon$, *where* ϵ *is a positive number.*
Then, system (18) possesses the convergence property.

Proof. System (18) satisfies that conditions of the theorem from [12]. Therefore, it has a unique bounded solution $\eta(t)$ that attracts all solutions $x(t;t_0;x_0)$ of system (18). Since all solutions of (12) are the solutions to (18), they are also attracted to $\eta(t)$. The set $\mathcal{M}$ from Definition 1 is defined by $\mathrm{rank} A = \mathrm{rank}(A|F(x_0,t_0))$ and, in virtue of $a_r(x,t) = \det B_{22}(Q^{-1}x,t)/\det(PQ) \neq 0$, is nonempty.

□

Let us give an auxiliary statement from [17] for a nonlinear system with a distinguished linear part.

Theorem 4. *Let there be given a system*

$$A\dot{x}(t) + Bx(t) = \phi(t,x), \tag{19}$$

and the following conditions be satisfied:
1. The matrix pencil $\lambda A + B$ *satisfies the rank-degree criterion;*
2. All roots $\lambda_1, \lambda_2, \ldots, \lambda_r$ *of the polynomial* $\det(\lambda A + B)$ *have nonzero real parts;*
3. $\phi(t,y) \in \mathbf{C}^1(I_t \times \|y\| < \infty)$ *u* $sup_t\|\phi(t,0)\| = \gamma < \infty$;
4. The Lipschitz condition holds: $\|\phi(t,y) - \phi(t,z)\| \leq L\|y - z\|\,\forall y,z \in \mathbf{R}^{\varrho}$, *and* L *is sufficiently small.*
Then:

1. There exists a solution $\eta = \eta(t)$ to (19), defined and bounded on I_t, and $\eta(t) = \eta(t+\omega)$ if $\phi(x,t) = \phi(x,t+\omega)$;
2. System (19) possesses the convergence property if $\mathrm{Re}\lambda_j < 0, (j = 1, \dots, r)$.

Theorems similar to Theorems 1, 2, 4 for the infinite-dimensional case can be found in [18, 19].

5. Investigation of the Hydraulic Circuit. Numerical Experiment

Let us establish some qualitative properties of system (7).

Lemma 6. *System (7) has index 2 if the matrix S_1 does not depend on the components of $P(t)$.*

Proof. Consider the product

$$Y_1 \circ \Omega(X,P) = \begin{pmatrix} E_n & 0 \\ \mathcal{A}_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{P} \end{pmatrix} + \begin{pmatrix} R^{-1}(t)S_0 & -R^{-1}(t)\mathcal{A}_1^\top \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ P \end{pmatrix} +$$

$$+ \begin{pmatrix} R^{-1}(t)S_1 \left|X(t)\right| X(t) \\ 0 \end{pmatrix}, \; Y_1 = \begin{pmatrix} R^{-1}(t) & 0 \\ 0 & (d/dt)E_m \end{pmatrix}, \tag{20}$$

where $\Omega(X,P)$ is an operator of system (7). If S_1 in (7) does not depend on P, then, transforming (20) by means of the operator $Y_2 = \begin{pmatrix} E_n & 0 \\ -(d/dt)\mathcal{A}_1 & (d/dt)E_m \end{pmatrix}$, we get

$$Y_2 \circ Y_1 \circ \Omega(x,p) = \begin{pmatrix} E_n & 0 \\ \tilde{U}(x) & \mathcal{A}_1 R^{-1}(t)\mathcal{A}_1^\top \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{P} \end{pmatrix} + \cdots ,$$

where $\tilde{U}(x)$ is some block of appropriate dimension. The matrix $\mathcal{A}_1$ is full rank if the matrix $R^{-1}(t)$ is diagonal with positive elements. Therefore, $\mathcal{A}_1 R^{-1}(t)\mathcal{A}_1^\top$ is nonsingular for all $t \in T$. The product $Y_2 \circ Y_1$ is the second order differential operator.

□

Below, we will need the following statement.

Theorem 5. *Let the system*

$$\mathcal{S}(t)\dot{y} + \Gamma(y, \mathcal{W}y, t) = 0, \; t \in T = [\alpha, \beta], \tag{21}$$

where $\mathcal{W}y = \int\limits_0^t \tilde{K}(t,\tau,y(\tau))d\tau$ *is the Volterra operator, satisfy the conditions:*
1. $\mathcal{S}(t) \in \mathbf{C}^2(T), \; \Gamma(y,t,z) \in \mathbf{C}^2(\{y : \|a-y\| \le \rho_1\} \times \{z : \|z\| \le \rho_2\} \times T, \; \rho_1, \rho_2 > 0),$ $a \in \mathbf{R}^\kappa, \; \tilde{K}(t,\tau,y) \in \mathbf{C}^1(T \times T \times \{y : \|a-y\| \le \rho_1\})$;
2. $\mathrm{rank}\, \mathcal{S}(t) = const, \; t \in [\alpha, \alpha+\rho_1]$;
3. $\mathrm{rank}\, \mathcal{S}(\alpha) = \mathrm{rank}\, \{\mathcal{S}(\alpha)|\Gamma(a,0,\alpha)\}$;
4. The polynomial $\lambda\mathcal{S}(\alpha) + \mathcal{D}(a,0,\alpha)$, where $\mathcal{D}(y,z,t) = \partial\Gamma(y,z,t)/\partial y$, satisfies the rank-degree criterion.
Then:
1. the solution $y(t) \in \mathbf{C}^1[\alpha, \alpha+\epsilon]$ to system (21) with the initial data $y(\alpha) = a$ is defined

on some segment $[\alpha, \alpha+\epsilon], \epsilon > 0$*;*
2. For the sufficiently small $h \leq h_0$*, there exist solutions to the difference scheme*

$$\mathcal{S}(t_{i+1})\frac{y_{i+1}-y_i}{h} + \Gamma\left(t_{i+1}, y_{i+1}, h\sum_{j=0}^{i}\tilde{K}(t_j, \epsilon_j, y_j)\right) = 0,\ y_0 = a, \tag{22}$$

where $h = \epsilon/N,\ t_{i+1} = \alpha + ih,\ i = \overline{0, N-1},\ \tau_j = \alpha + jh$*, and the following estimation holds:*

$$\|y_i - y(t_i)\| \leq Ch,\ C = const > 0.$$

More over, (22) can be replaced by the non-iterative scheme

$$\mathcal{S}(t_{i+1})\frac{w_{i+1}-w_i}{h} + \Gamma(\xi_i) + \mathcal{D}(\xi_i)(w_{i+1}-w_i) = 0,\ \xi_i = \left(t_{i+1}, w_i, h\sum_{j=0}^{i}\tilde{K}(t_j, \tau_j, w_j\right), \tag{23}$$

with the estimation $\|w_i - y(t_i)\| \leq C_1 h,\ C_1 = const > 0.$

Theorem 5 is a special case of statements proved earlier in [20].

Lemma 7. *Let system (7)* $H(t) \equiv 0,\ P^*(t) \equiv 0,\ Q_1(t) \equiv 0,\ t \in I_t$*, have no regulator components. Then (7) possesses the convergence property.*

Proof. Define the set $\mathcal{M}$ from Definition 1. It is clear that under the current conditions $\mathcal{A}_1 X(t_0) = 0$. Moreover, system (7) does not satisfy the forth condition of Theorem 5, meanwhile, system $Y_1 \circ \Omega(X, P) = 0$ does. By direct calculation, it can be shown that $\det[\lambda\mathcal{S}(\alpha) + \mathcal{D}(a, 0, \alpha)] = [\mathcal{A}_1 R^{-1}(t)\mathcal{A}_1^\top]\lambda^n + \cdots$, and the forth condition is satisfied for all values of t. To satisfy the condition 3, it is necessary and sufficient that the following equality holds at the point t_0

$$[\mathcal{A}_1 R^{-1}(t)\mathcal{A}_1^\top]P(t) = -\mathcal{A}_1 R^{-1}(t) S_1 \left|X(t)\right| X(t). \tag{24}$$

The set of solutions to $\mathcal{A}_1 X(t_0) = 0$ is nonempty. Since the matrix $\mathcal{A}_1 R^{-1}(t)\mathcal{A}_1^\top$ is nonsingular, for any $X(t_0) = 0$ there exists an allowable initial value $P(t_0) = 0$. Hence, the set $\mathcal{M}$ is nonempty, and, by virtue of Theorem 5, there exists a neighbourhood of t_0 where the solution to (7) is defined for allowable initial values.

Let $y(t) = \begin{pmatrix} X(t) \\ P(t) \end{pmatrix}$ be a solution to (7). Consider the inner product

$$(y, \Omega(X, P)) = (1/2)\sum_{i=1}^{n} r_i(t)\frac{d}{dt}[x_i^2(t)] + \sum_{i=1}^{n}\left[s_{0.i}x_i^2(t) + s_{1,i}x_i^2(t)\left|x_i(t)\right|\right] = 0. \tag{25}$$

Here, we took into consideration that $R(t)$ is diagonal and $-(X, \mathcal{A}_1^\top P) + (P, \mathcal{A}_1 X) \equiv 0$. Assume that (7) has a nonzero bounded solution $\eta(t)$ defined on I_t. Equation (25) implies that for any nonzero solution

$$\sum_{i=1}^{n} r_i(t)\frac{d}{dt}[x_i^2(t)] < 0. \tag{26}$$

Therefore, when $t \to -\infty$ we have $\|\eta(t)\| \to \infty$. We arrived at a contradiction. The nonzero solution is a unique bounded solution on I_t. The validity of the lemma follows from (24) and (26).

□

Let us now present results of numerical experiments. It follows from (23) that $w_{i+1} = M^{-1}(\xi_i)\Big[[\mathcal{S}(t_{i+1}) + h\mathcal{D}(\xi_i)]w_i - h\Gamma(\xi_i)\Big]$, where $M(\xi_i) = \mathcal{S}(t_{i+1}) + h\mathcal{D}(\xi_i)$. The forth condition of Theorem 5 implies that matrix $M(\xi_i)$ is invertible for the sufficiently small h. In Example 1, set $\theta_1 = 75$, $\theta_3 = 9$, $R = \operatorname{diag}\{1;1;1;1;1;1\}$, $X_0 = (x_{10}; x_{20}; \ldots, x_{60})^\top = (65;45;20;65;65;65)^\top$, $p^* = (p_5^*; p_6^*)^\top = (200;0,035)^\top$, $P_0 = (p_{10}; p_{20}; p_{30}; p_{40})^\top = (185;170;160;120)^\top$, $q(t) \equiv 0$,
$S_0 = 0, S_1 = \operatorname{diag}\{s_{11}; s_{12}; \ldots; s_{16}\} = \operatorname{diag}\{0,0005;0,0014;0,0444;0,1825;0,4508;0,3336\}$.

In scheme (23), we set $h = 0,001$; $t \in [0..50]$.

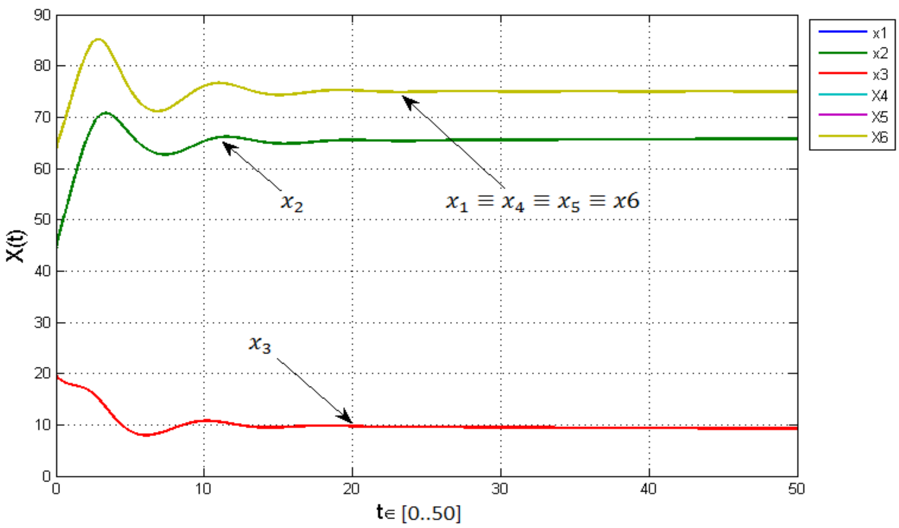

Fig 2. Diagram for the flow rates in pipelines with regulators

Fig. 2 shows that the regulators brought the flow rates to the given values.

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ИССЛЕДОВАНИЕ МОДЕЛЕЙ НЕСТАЦИОНАРНЫХ ГИДРАВЛИЧЕСКИХ ЦЕПЕЙ НА ОСНОВЕ ТЕОРИИ ВЫРОЖДЕННЫХ СИСТЕМ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Е.В. Чистякова, Нгуен Дык Банг

Анализ сложных гидравлических систем, электрических цепей, электронных схем, химических реакций и т.д. часто приводит к необходимости решать системы взаимосвязанных дифференциальных и алгебраических уравнений. Если процесс обладает последействием, то такие системы могут включать интегральные уравнения. Данная работа посвящена моделированию гидравлических цепей с помощью вырожденных интегро-дифференциальных уравнений. Приводятся теоретические результаты, с помощью которых изучаются качественные свойства рассматриваемых систем и строятся эффективные численные методы. В работе рассмотрена модель гидравлической цепи пароводяного тракта прямоточного котла, предложен численный метод решения. Экспериментальные результаты показали, что теория вырожденных систем интегро-дифференциальных уравнений хорошо работает при моделировании процессов, протекающих в гидравлических системах.

Keywords: дифференциально-алгебрачиеские уравнения; интегро-алгебраические уравнения; гидравлические цепи; индекс, численные методы.

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