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$\mathcal{L}$-STABILITY OF NONLINEAR SYSTEMS REPRESENTED BY STATE MODELS

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Stability theory plays a key role in systems theory and engineering. The stability of equilibrium points is usually considered within the framework of the stability theory developed by the Russian mathematician and mechanic A.M. Lyapunov (1857–1918), who laid its foundations and gave it its name. Nowadays, the point of view on stability has become very widespread, as stability in relation to disturbance of the input signal. The research is based on the space-state approach for modelling nonlinear dynamic systems and an alternative "input-output" approach. The input-output model is implemented without explicit knowledge of the internal structure determined by the equation of state. The system is considered as a "black box", which is accessed only through the input and output terminals ports. The concept of stability in terms of "input-output" is based on the definition of $\mathcal{L}$-stability of a nonlinear system, the method of Lyapunov functions and its generalization to the case of nonlinear dynamical systems. The interpretation of the problem on accumulation of perturbations is reduced to the problem on finding the norm of an operator, which makes it possible to expand the range of models under study, depending on the space in which the input and output signals act.

Keywords: dynamical system; $\mathcal{L}$-stability; exponential stability; causality; gain factor.

Introduction

In 1880, the problems of the qualitative theory of differential equations were formulated by the French mathematician and mechanic A. Poincaré. Over time, with the development of science and technology, problems about the stability of not only equilibrium positions, but also the movement of the system as a whole, arose. The crisis caused by the unstable operation of centrifugal regulators installed on steam machines was the reason for the emergence of investigations by J.C. Maxwell (1868), I.A. Vyshnegradsky (1876–1877) A. Stodol [1] and others, in which certain issues of the theory of regulation were considered depending on the establishment of criteria for stability of motion.

The papers of E.J. Routh (1877–1884) [2], N.E. Zhukovsky (1882) [3] present forward problems and methods for their solution and are devoted to general issues of motion stability. In the papers of that time, when analyzing stable properties, the authors proceeded from the linearized equations of perturbed motion without taking into account the influence.

In this direction, the starting point for all researches is the classical book of A.M. Lyapunov "General problem of the stability of motion", which appeared in Russia in 1892 [4]. In Lyapunov theory, the central issue is the stability of equilibrium points.

The purpose of this research is to study generalizations of the main Lyapunov theorems to a wider class of systems of differential equations and their application for use in problems of analyzing the stability of various processes and phenomena in their mathematical modelling.

1. Stability in Terms of "Input-Output"

Consider a system in which the connection between inputs and outputs is represented by the equality $y = Hu$, where H is some operator or some mapping that defines y in terms of u. The input u belongs to the signal space and maps the time interval $[0, \infty)$ to the Euclidean space $\mathbb{R}^m$. As examples of such spaces, we note the space of piecewise continuous bounded functions, or the space of piecewise continuous square-integrable functions.

To measure the signal intensity, we introduce the norm of the function u satisfying the following properties:

– the signal norm is zero if the signal is identically equal zero, otherwise;

– the norm takes strictly positive values;

– scaling the signal leads to the corresponding scaling of its norm, i.e. $\|au\| = a\|u\|$ for any positive constant and any signal;

– the norm satisfies the triangle inequality $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$.

For example, in the case of the space of piecewise continuous square-integrable functions, the norm is defined by the equality

$$\|u\|_{\mathcal{L}_2} = \sqrt{\int\limits_0^\infty u^T(t)u(t)dt} < \infty$$

and this space is denoted by $\mathcal{L}_2^m$. In general, the space $\mathcal{L}_p^m$, $1 \leq p \leq \infty$, is defined as the set of all piecewise continuous functions u such that

$$\|u\|_{\mathcal{L}_2} = \left(\int\limits_0^\infty \|u(t)\|^p dt\right)^{1/p} < \infty.$$

In $\mathcal{L}_p^m$, the subscript p corresponds to the type of the p-norm used to define the space, and the superscript m corresponds to the dimension of the signal u. If it is clear from the context which subscript or superscript is considered, then we omit it and denote the corresponding space $\mathcal{L}_p$, $\mathcal{L}^m$ or $\mathcal{L}$. In order to distinguish the norm of the vector u in the space $\mathcal{L}$ and the norm of the vector $u(t)$ in the space $\mathbb{R}^m$, we write the first one in the form $\| \cdot \|_{L^2}$.

If the input $u \in \mathcal{L}^m$ has "satisfactory behavior", then the question arises whether it has the same "satisfactory behavior" in the sense that $y \in \mathcal{L}^q$, where $\mathcal{L}^q$ is a space similar to the space $\mathcal{L}^m$, but in the general case the number of output variables q may differ from the number of input variables m. A system is characterized as a stable one, if the system has the property that in the case of "satisfactory behavior" of the input action, it generates an output signal with "satisfactory behavior". However, it is impossible to define H as a mapping from $\mathcal{L}^m$ to $\mathcal{L}^q$, since one can obtain unstable systems in which the input $u \in \mathcal{L}^m$ can generate the output y, which does not belong to $\mathcal{L}^q$. Therefore, H is usually defined as a mapping from the extended space $\mathcal{L}_e^m$ to $\mathcal{L}_e^q$, where

$$\mathcal{L}_e^m = \{u | u_\tau \in \mathcal{L}^m, \ \forall \tau \in [0, \infty)\}$$

and u_τ is the truncation function:

$$u_\tau = \begin{cases} u(t), & 0 \leq t \leq \tau, \\ 0, & t > \tau. \end{cases}$$

The extended space $\mathcal{L}_e^m$ is a linear space containing the unextended space $\mathcal{L}^m$. This allows to deal with unbounded signals. For example, the signal $u(t) = t$ does not belong to the space $\mathcal{L}_\infty$, but its truncation

$$u_\tau = \begin{cases} t, & 0 \le t \le \tau, \\ 0, & t > \tau, \end{cases}$$

belongs to the extended space. The mapping

$$H: \ \mathcal{L}_e^m \to \mathcal{L}_e^q$$

is called casual or non-anticipatory if the output value $(Hu)(t)$ at any time t depends only on entering the past times, including t. In other words,

$$(Hu)_\tau = (Hu_\tau)_\tau.$$

Causality is a property inherent in dynamical systems presented in the form of a state space model. Since the spaces of input and output signals are defined, we can define the stability of the "input-output".

Definition 1. *The mapping $H: \ \mathcal{L}_e^m \to \mathcal{L}_e^q$ is called $\mathcal{L}$-stable if there exist a $\mathcal{K}$-function α defined on $[0, \infty)$ and a nonnegative constant β such that*

$$\|(Hu)_\tau\|_\mathcal{L} \le \alpha\left(\|u_\tau\|_\mathcal{L}\right) + \beta \tag{1}$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$. This mapping H is called $\mathcal{L}$-stable with a finite gain if there exist nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_\mathcal{L} \le \gamma\left(\|u_\tau\|_\mathcal{L}\right) + \beta \tag{2}$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$.

The constant β in (1) and (2) is called the displacement term. This term is presented in the definition, since it is necessary to take into account situations when the output of Hu in the system is not equal to zero for $u = 0^3$. If inequality (2) is true, then it is interesting to obtain the smallest value of γ for which there exists β under which (2) holds. In the situations where such a value γ can be determined, this value is called a system gain factor. If inequality (2) holds under some $\gamma \ge 0$, then the system has $\mathcal{L}$-gain factor less than or equal to γ.

For a casual, $\mathcal{L}$-stable system, it can be shown that

$$u \in \mathcal{L}^m \ \Rightarrow Hu \in \mathcal{L}^q$$

and

$$\|(Hu)_\tau\|_\mathcal{L} \le \alpha\left(\|u_\tau\|_\mathcal{L}\right) + \beta, \ \forall u \in \mathcal{L}^m.$$

For a casual $\mathcal{L}$-stable system with a finite gain factor, the previous inequality takes the following form:

$$\|(Hu)_\tau\|_\mathcal{L} \le \gamma\left(\|u_\tau\|_\mathcal{L}\right) + \beta, \ \forall u \in \mathcal{L}^m.$$

The definition of $\mathcal{L}_\infty$-stability coincides with the definition of stability of a "bounded input-bounded output" type system: if the system is $\mathcal{L}_\infty$-stable, then, for any bounded input $u(t)$, the output of $Hu(t)$ is also bounded.

Example 1. Let $h : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a memory-less function, which is possibly time-dependent and can be considered as an operator H, matching the input signal $u(t)$ to the output signal $y(t) = h(t, u(t))$. We use this operator to illustrate the definition of $\mathcal{L}$-stability.

Solution

Assume that

$$h(t) = a + b\ \operatorname{th} cu = a + b\frac{e^{cu} - e^{-cu}}{e^{cu} + e^{-cu}},$$

where a, b, c are some non-negative constants. Considering that

$$h'(u) = \frac{4bc}{(e^{cu} + e^{-cu})^2} \le bc, \quad \forall\ u \in \mathbb{R}$$

we have

$$|h(u)| \le a + bc\,|u|, \quad \forall u \in \mathbb{R}.$$

Consequently, the operator H is $\mathcal{L}_\infty$-stable with a finite gain factor $\gamma = bc$, $\beta = a$. In addition, if $a = 0$, then, for any $p \in [0, \infty)$,

$$\int_0^\infty |h(u(t))|^p\, dt \le (bc)^p \int_0^\infty |u(t)|^p\, dt.$$

Thus, for any $p \in [0, \infty)$, the operator H is $\mathcal{L}_\infty$-stable with a finite gain factor $\gamma = bc$, and the displacement is zero. Let h be a time-dependent function satisfying the condition

$$|h(t, u)| \le a\,|u|,\ \forall t \ge 0,\ \forall u \in \mathbb{R},$$

where a is some positive constant. For any $p \in [0, \infty)$, the operator H is $\mathcal{L}_p$-stable with a finite gain factor $\gamma = a$, and the displacement is zero. Finally, assume that

$$h(u) = u^2.$$

Since

$$\sup_{t\ge 0} |h(u(t))| \le \left(\sup_{t\ge 0} |u(t)|\right)^2,$$

then the operator H is $\mathcal{L}_\infty$-stable and displacement is zero. This operator is not $\mathcal{L}_\infty$-stable with a finite gain factor, since the function $h(u) = u^2$ cannot be subject to restrictions of the form $|h(u)| \le \gamma\,|u| + \beta$ for all $u \in \mathbb{R}$.

One of the drawbacks of Definition 1 is that it explicitly imposes the condition of inequalities (1) and (2) for any input signals from the space $\mathcal{L}^m$. These requirements exclude from consideration the systems in which the input-output relations are defined only on a subset of the input signal space. Let us give an example motivating the need to introduce a definition of a new concept of $\mathcal{L}$-stability for a small input signal.

Example 2. Consider the system with one input and one output, which is defined by nonlinearity

$$y = \operatorname{tg} u.$$

Solution

The output $y(t)$ is defined only for input signals satisfying the inequality

$$|u(t)| < \frac{\pi}{2}, \quad \forall t \geq 0.$$

Thus, the system is not $\mathcal{L}_\infty$-stable in the sense of Definition 1. However, if we are restricted to consideration of the input signals $u(t)$ satisfying

$$|u| \leq r < \frac{\pi}{2},$$

then

$$|y| \leq \left(\frac{\operatorname{tg} r}{r}\right)|u|$$

and the system satisfies the inequality

$$\|y\|_{\mathcal{L}_p} \leq \left(\frac{\operatorname{tg} r}{r}\right)\|u\|_{\mathcal{L}_p}$$

for any $u \in \mathcal{L}_p$ such that $|u(t)| \leq r$ for all $t \geq 0$, where $p \in [0, \infty)$. In the space $\mathcal{L}_\infty$ the inequality $|u(t)| \leq r$ implies $\|u(t)\|_{\mathcal{L}_\infty} \leq r$, i.e. the above inequality holds only for input signals with small norm. However, in the case of considering other spaces $\mathcal{L}_p$, $p < \infty$, the instantaneous boundedness of the value of $|u(t)|$ does not imply the boundedness of the corresponding norm of this input signal.

Definition 2. *The mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is called $\mathcal{L}$-stable at a small input signal (respectively, $\mathcal{L}$-stable at a small input signal with a finite gain factor) if there exists a positive constant r such that inequality* (1) *(respectively, inequality* (2)*) holds for all $u \in \mathcal{L}_e^m$, $\sup_{0 \leq t \leq \tau} \|u(t)\| \leq r$.*

2. $\mathcal{L}$-Stability of State Models

In this section, we show how the means of establishing stability proposed by Lyapunov's theory can be used to prove the $\mathcal{L}$-stability of nonlinear systems represented by state models.

Consider the system

$$\dot{x} = f(t, x, u), \quad x(0) = x_0, \tag{3}$$

$$y = h(t, x, u), \tag{4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^q$, the function $f : [0, \infty) \times D \times D_u \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz by (x, u), $D \subset \mathbb{R}^n$ is the open area containing the origin $x = 0$ and $x = 0$ и $D_u \subset \mathbb{R}^m$ is the area containing $u = 0$. For each fixed $x_0 \in D$, state model (3) – (4) defines an operator H, assigning to each input $u(t)$ the corresponding output $y(t)$. Suppose that $x = 0$ is the equilibrium point of the free system

$$\dot{x} = f(t, x, 0). \tag{5}$$

Theorem 1. *Consider system* (3) – (4) *and choose $r > 0$ and $r_u > 0$ such that $\{\|x\| \leq r\} \subset D$ and $\{\|x\| \leq r_u\} \subset D_u$. Suppose that*

1) $x = 0$ is an exponentially stable equilibrium point of (5) and there exists a Lyapunov function $V(t, x)$ satisfying the inequalities

$$c_1\|x\|^2 \leq V(t, x) \leq c_2\|x\|^2, \tag{6}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, 0) \leq -c_3\|x\|^2, \tag{7}$$

$$\left\|\frac{\partial V}{\partial t}\right\| \leq c_4\|x\|^2 \tag{8}$$

for all $(t, x) \in [0,\ \infty) \times D$ and some positive constants c_1, c_2, c_3 and c_4.

2) The functions f and h satisfy the inequalities

$$\|f(t, x, u) - f(t, x, 0)\| \leq L\|u\|, \tag{9}$$

$$h(t, x, u) \leq \eta_1\|x\| + \eta_2\|u\| \tag{10}$$

for all $(t, x, u) \in [0,\ \infty) \times D \times D_u$ and some non-negative constants L, η_1, η_2.

Then, for each x_0 such that $x_0 \leq r\sqrt{c_1/c_2}$, system (3) – (4) is $\mathcal{L}_p$-stable at a small input signal with a finite gain factor for all $p \in [0,\ \infty]$. In particular, for every $u \in \mathcal{L}_p$ such that

$$\sup_{0 \leq t \leq \tau} u \leq \min\left\{r_u, \frac{c_1 c_3 r}{c_2 c_4 L}\right\},$$

the output of the system $y(t)$ satisfies the estimate

$$\|y_\tau\|_{\mathcal{L}_p} \leq \gamma\|u\|_{\mathcal{L}_p} + \beta \tag{11}$$

for all $\tau \in [0,\ \infty)$, where

$$\gamma = \eta_2 + \frac{\eta_1 c_2 c_4 L}{c_1 c_3}$$

$$\beta = \eta_1\|x_0\|\sqrt{\frac{c_2}{c_1}}\rho, \quad where \quad \rho = \begin{cases} 1 & if \ \ p = \infty, \\ \left(\frac{2c_2}{c_3 p}\right)^{\frac{1}{p}} & if \ \ p \in [1, \infty). \end{cases}$$

In addition, if the origin is globally exponentially stable and all assumptions are satisfied globally (for $D = \mathbb{R}^n$ and $D_u = \mathbb{R}^m$), then for each $x_0 \in \mathbb{R}^n$ system (3) – (4) is $\mathcal{L}_p$-stable with a finite gain factor for all $p \in [0,\ \infty]$.

The proof of Theorem 1 is presented in [5].

Using the Lyapunov converse theorem [6], we can show that there exists a Lyapunov function satisfying (6) – (8). Then the following corollary can be formulated.

Corollary 1. *Suppose that, in some neighborhood of the point $(x, u) = (0,\ 0)$, the function $f(t,\ x,\ u)$ is continuously differentiable, the Jacobi matrices $\left[\frac{\partial f}{\partial x}\right]$ and $\left[\frac{\partial f}{\partial u}\right]$ are uniformly bounded in t and $h(t, x, u)$ satisfies (10). If the origin $x = 0$ is an exponentially stable equilibrium point of (5), then there exists a constant $r_0 > 0$ such that for any x_0, $\|x_0\| < r_0$ system (3) – (4) is $\mathcal{L}_p$-stable at a small input with the finite gain factor for all $p \in [0,\ \infty]$. Moreover, if all assumptions are fulfilled globally and the origin is a globally exponentially stable equilibrium point of (5), then for any $x_0 \in \mathbb{R}^n$ system (3) – (4) is $\mathcal{L}_p$-stable with finite gain factor for all $p \in [0,\ \infty]$.*

In the next example, we illustrate how the method of Lyapunov functions is generalized to the case of the second-order systems with one input and one output.

Example 3. Consider the system

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = -x_1 - x_2 = a \ \operatorname{th} x_1 + u, \\ y = x_1, \end{cases}$$

where a is some non-negative constant. We use

$$V(x) = x^T P x = p_{11} x_1^2 + 2p_{12} x_1 x_2 + p_{22} x_2^2$$

as the Lyapunov function for a free system:

$$\dot{V} = -2p_{12}\left(x_1^2 + a x_1 \operatorname{th} x_1\right) + 2\left(p_{11} - p_{12} - p_{22}\right) x_1 x_2 - 2a p_{22} x_2 \ \operatorname{th} x_1 - 2\left(p_{22} - p_{21}\right) x_2^2.$$

To get rid of the cross term $x_1 x_2$, we assume $p_{11} = p_{12} + p_{22}$. Then, for $p_{22} = 2p_{12} = 1$, the matrix P is positive definite and

$$\dot{V} = -x_1^2 - x_2^2 - a x_1 \operatorname{th} x_1 - 2a x_2 \operatorname{th} x_1.$$

Taking into account that $x_1 \operatorname{th} x_1 \geq 0$ for all $x_1 \in \mathbb{R}$ we obtain

$$\dot{V} \leq -\|x\|_2^2 + 2a\,|x_1| \cdot |x_2| \leq -\left(1 - a\right)\|x\|_2^2.$$

Thus, for all $a < 1$ the function V satisfies (6) – (8) globally with

$$c_1 = \lambda_{min}\left(P\right), \quad c_2 = \lambda_{max}\left(P\right), \quad c_3 = 1 - a, \quad c_4 = 2\|P\|_2 = 2\lambda_{max}\left(P\right).$$

The functions f and h globally satisfy (9) and (10) with $L = \eta_1 = 1$, $\eta_2 = 0$. Therefore, for all $x_0 \in \mathbb{R}^2$ and $p \in [1, \infty]$, the system is $\mathcal{L}_p$-stable with a finite gain factor.

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$\mathcal{L}$-УСТОЙЧИВОСТЬ НЕЛИНЕЙНЫХ СИСТЕМ, ПРЕДСТАВЛЕННЫХ МОДЕЛЯМИ СОСТОЯНИЯ

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Теория устойчивости играет ключевую роль в теории систем и инженерных науках. Устойчивость точек равновесия обычно рассматривается в рамках теории устойчивости, разработанной русским математиком и механиком А.М. Ляпуновым (1857–1918), заложившим ее основы и давшим ей имя. В настоящее время стала очень распространенной точка зрения на устойчивость, как устойчивость по отношению к возмущению входного сигнала. В основу исследования положен подход пространства-состояния для моделирования нелинейных динамических систем и альтернативный подход «вход-выход». Модель «вход-выход» реализуется без явного знания внутренней структуры, определяемой уравнением состояния. Система рассматривается как «черный ящик», доступ к которому осуществляется только через входные и выходные терминалы порты. В основу концепции устойчивости в терминах «вход-выход» положено определение $\mathcal{L}$-устойчивости нелинейной системы, метод функций Ляпунова и его обобщение на случай нелинейных динамических систем. Трактовка задачи о накоплении возмущений сводится к задаче отыскания нормы оператора, что позволяет расширить круг исследуемых моделей в зависимости от пространства, в котором действуют входные и выходные сигналы.

Ключевые слова: динамическая система; $\mathcal{L}$-устойчивость; экспоненциальная устойчивость; казуальность; коэффициент усиления.

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