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INVARIANT SPACES OF OSKOLKOV STOCHASTIC LINEAR EQUATIONS ON THE MANIFOLD

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The Oskolkov equation is obtained from the Oskolkov system of equations describing the dynamics of a viscoelastic fluid, after stopping one of the spatial variables and introducing a stream function. The article considers a stochastic analogue of the linear Oskolkov equation for plane-parallel flows in spaces of differential forms defined on a smooth compact oriented manifold without boundary. In these Hilbert spaces, spaces of random K-variables and K-“noises” are constructed, and the question of the stability of solutions of the Oskolkov linear equation in the constructed spaces is solved in terms of stable and unstable invariant spaces and exponential dichotomies of solutions. Oskolkov stochastic linear equation is considered as a special case of a stochastic linear Sobolev-type equation, where the Nelson–Glicklich derivative is taken as the derivative, and a random process acts as the unknown. The existence of stable and unstable invariant spaces is shown for different values of the parameters entering into the Oskolkov equation.

Keywords: Sobolev-type equations; differential forms; Nelson-Glicklich derivative; invariant spaces.

Introduction

Consider the Oskolkov equation

$$(\lambda-\Delta)\Delta\dot{\psi}=\nu\Delta^2\psi-\frac{\partial(\psi,\Delta\psi)}{(x_1,x_2)}. \qquad (1)$$

Equation (1) is a model of the flow of a viscous and elastic incompressible fluid [1], in which the parameter ν is responsible for the viscous properties of the fluid. The parameter λ that determines the elastic properties of a fluid can take positive and negative values [2]. In [3–5], the solvability of the Cauchy-Dirichlet problem for the Oskolkov equation (1) was considered, and in [6] for the linear Oskolkov equation

$$(\lambda-\Delta)\Delta\dot{\psi}=\nu\Delta^2\psi. \qquad (2)$$

In [7], the problem of stability of solutions to equation (2) was solved in terms of exponential dichotomies, and in paper [8], the problem of stability of solutions in a neighborhood of the zero point of equation (1) was solved in terms of invariant manifolds.

This article discusses the stability of the stochastic linear Oskolkov equation on manifold that has no boundary. To solve this issue, we use equation (2) as an equation of the following form:

$$L\overset{o}{\eta}=M\eta, \qquad (3)$$

where $\overset{o}{\eta}$ derivative in the sense [9] of the sought-for random process $\eta=\eta(t)$. The number of works devoted to the study of equations of the form (3) is quite large at the present time (see, for example, [10–13]), in which this equation was considered in various aspects. The present work is closest to [14] and [15], in which we study the solvability and stability of the Barenblatt–Zheltov–Kochina stochastic equation on a manifold.

The article contains four parts. The first section is the introduction, the fourth section is the bibliography. The second point is dedicated to deals with spaces of q-forms defined on a manifold that has no boundary, recalls the notions of a random variable, stochastic process, Nelson–Gliklikh derivative, constructs spaces of random K-variables and K-“noises”. The third point contains a description of the invariant spaces of the Oskolkov stochastic equation.

1. Spaces of "noises" on a manifold

Consider *n*-dimensional manifold Ω that has no boundary. Let it have the properties of connectedness, compactness and smoothness. Consider spaces of smooth shapes $E^q = E^q(M), 0 \le q \le n$ on Ω, where the scalar products are defined by the following formulas:

$$(a,b)_0 = \int_\Omega a\Lambda * b,\ (a,b)_2 = (\Delta a, \Delta b)_0 + (\Delta a, b)_0 + (a,b)_0,$$

$\Delta = d\delta + \delta d$ is the Laplace–Beltrami operator, $\delta = (-1)^{n(q+1)+1} * d *$, where $*$ is the Hodge operator associating a differential form E^q with a differential form E^{n-q}, d is the outer differentiation operator. The spectrum $\sigma(\Delta) = \{\alpha_k\}$ of the operator Δ is positive discrete, and $+\infty$ is the point of its condensation. Denote by H_0^q and H_2^q the completions of the lineal E^q with respect to the norms $\|\cdot\|_0$ and $\|\cdot\|_2$. The basis in Hilbert spaces H_l^q is the sequence of eigenfunctions $\{\varphi_k\}$ of the operator Δ orthonormalized by the norms $\|\cdot\|_l$, $l = 0, 2$.

Next, we turn to the construction of spaces of random K - variables and K -"noises" in H_l^q. Let $\Omega = (\Omega, A, P)$ be a full probability space. We define a random variable as a mapping $\xi : \Omega \to R$ and stochastic process as mapping $\eta : \Im \times \Omega \to R$ (where $\Im$ is a certain interval from R, a function $\eta = \eta(\cdot, \omega)$ is a trajectory of the stochastic process). If almost all trajectories of a random process are continuous then such a process is called continuous.

L_2 is the set of random variables ξ for which the variance D is finite and the mathematical expectation E is zero, and CL_2 is the set of continuous stochastic processes η. We fix $t \in \Im$, let $E_t^\eta = E\left(| N_{t^\eta}\right)$, N_{t^η} the σ-algebra generated by the random variable η. By the derivative in the sense [9] $\overset{\circ}{\eta}$ of the stochastic process $\eta \in CL_2$ в $t \in \Im$ we mean a limit (if it exists)

$$\overset{\circ}{\eta} = \frac{1}{2}\left(\lim_{\Delta t \to 0+} E_t^\eta \left(\frac{\eta(t+\Delta t, \cdot) - \eta(t, \cdot)}{\eta}\right) + \lim_{\Delta t \to 0+} E_t^\eta \left(\frac{\eta(t, \cdot) - \eta(t - \Delta t, \cdot)}{\eta}\right)\right).$$

Denote by C^1L_2 the space of stochastic processes whose trajectories are almost sure differentiable in the sense [9] on $\Im$. The spaces C^1L_2 are called spaces of differentiable "noises".

Let us introduce into consideration the space $H_l^q, l = 0, 2$ whose elements are random K -variables

$$\eta = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k .$$

The norm in this space is defined by the following formula:

$$\|\eta\|_{H_l^q} = \sum_{k=1}^{\infty} \lambda_k^2 D\xi_k ,$$

where $\{\xi_k\}$ is a sequence of random variables with bounded variance, $\{\varphi_k\}$ are the eigenfunctions of the operator Δ, orthonormalized by $(\cdot,\cdot)_l$, $l = 0, 2$, and $K = \{\lambda_k\}$ is a monotone sequence such that $\sum_{k=1}^{\infty} \lambda_k < +\infty$. Let $C(\Im; H_l^q)$ be the set of continuous stochastic K -processes

$$\eta(t) = \sum_{k=1}^{\infty} \lambda_k \eta_k(t) \varphi_k ,\ \eta_k \in CL_2 , \tag{4}$$

and $C^1(\Im; H_l^q)$ be the set of continuously Nelson–Gliklikh differentiable K -processes

$$\overset{\circ}{\eta}(t) = \sum_{k=1}^{\infty} \lambda_k \overset{\circ}{\eta}_k(t) \varphi_k ,\ \overset{\circ}{\eta}_k \in C^l L_2 , \tag{5}$$

if series (4) and (5) converge uniformly on $\Im \subset R$ ($\Im$ is compact set in R).

2. Stable and unstable invariant spaces

Let us define the operators

$$L = -(\lambda + \Delta)\Delta,\ M = \nu\Delta^2$$

and the equation (2) in the space H_0^q can be considered in the form

$$L\overset{o}{\eta} = M\eta\ . \tag{6}$$

The operators $L, M : H_0^q \to H_2^q$ have the properties of linearity and continuity, and the operator M is $(L,0)$-bounded operator.

By a solution to equation (6) we mean a stochastic K-process $\eta \in C^1(\Im; H_0^q)$ if, after substituting it into equation (6), we obtain the identity.

Definition 1. A set $P \in H_0^q$ such that the following conditions are satisfied:

(i) almost sure each trajectory of the solution $\eta = \eta(t)$ to equation (6) belongs to P;

(ii) for almost all $\eta_0 \in P$, there exists a solution to equation (6) satisfying the condition

$$\eta(0) = \eta_0 \tag{7}$$

is called a phase space of equation (6).

It was shown (see, for example, [14]) that the phase space of equation (3) is the image of the resolving group $U^t = \dfrac{1}{2\pi i}\int_\gamma (\mu L - M)^{-1} L e^{\mu t} d\mu$. Therefore, the following theorem is true.

Theorem 1. The set of the following form:

$$P = \begin{cases} H_0^q,\ \lambda \in \{\alpha_k\}, \\ \eta \in H_0^q : (\eta, \varphi_n)_0 = 0,\ \lambda = \alpha_n \end{cases} \tag{8}$$

is the phase space of equation (6)

If the solution to problem (6), (7) is $\eta \in C^1(\Im; I)$ for any $\eta_0 \in L$, $I \subset P$, then the set I called a invariant space of equation (6).

Definition 2. A set I_+ such that the following conditions are satisfied

(i) I_+ is an invariant space;

(ii) $\|\eta^1(t)\|_{H_0^q} \le N_1 e^{-m_1(s-t)} \|\eta^1(s)\|_{H_0^q}$, $s \ge t$, where positive constants N_1, m_1, $\eta^1 \in I_+$ for all $t \in R$

is called a stable invariant space of equation (6). A set I_- such that the following conditions are satisfied

(i) I_+ is an invariant space;

(ii) $\|\eta^2(t)\|_{H_0^q} \le N_2 e^{-m_2(t-s)} \|\eta^1(s)\|_{H_0^q}$, $t \ge s$, where positive constants N_2, m_2, $\eta^2 \in I_-$ for all $t \in R$

is called an unstable invariant space of equation (6).

Due to the fact that the relative spectrum of the operator M $\sigma^L(M) = \sigma_+^L(M) + \sigma_-^L(M)$, where

$$\sigma_+^L(M) = \left\{ \frac{-\nu\alpha_k}{\lambda + \alpha_k},\ \lambda > -\alpha_k \right\},\ \sigma_-^L(M) = \left\{ \frac{-\nu\alpha_k}{\lambda + \alpha_k},\ \lambda < -\alpha_k \right\},$$

and the results presented in [15] we obtain

Theorem 2. (i) The stable invariant space is set of the form (8) for $\nu > 0$ and $\lambda < 0$.

(ii) The stable invariant space is set of the form

$$I_+ = \left\{ \eta \in H_0^q : (\eta, \varphi_k)_0 = 0,\ \lambda > -\alpha_k \right\}$$

and the unstable invariant space is the set of the form

$$I_{-} = \left\{ \eta \in H_0^q : (\eta, \varphi_k)_0 = 0, \lambda < -\alpha_k \right\}$$

for $\nu > 0$ and $\lambda < 0$.

Remark 1. For $\nu > 0$ and $\lambda < 0$ there is an exponentially dichotomous behavior of solutions to the equation (6).

Conclusion

In the future, we intend to study the question on the solvability and stability of the stochastic analogue of semilinear equation (1). In addition, we intend to transfer all results for equation (1) to spaces of *q*-forms defined on a manifold with border.

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ИНВАРИАНТНЫЕ ПРОСТРАНСТВА СТОХАСТИЧЕСКОГО ЛИНЕЙНОГО УРАВНЕНИЯ ОСКОЛКОВА НА МНОГООБРАЗИИ

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Уравнение Осколкова получается из системы уравнений Осколкова, описывающей динамику вязкоупругой жидкости, после купирования одной из пространственных переменных и введения функции тока. В статье рассматривается стохастический аналог линейного уравнения Осколкова плоскопараллельных течений в пространствах дифференциальных форм, определенных на гладком компактном ориентированном многообразии без края. В данных гильбертовых пространствах строятся пространства случайных K-величин и K-«шумов» и решается вопрос об устойчивости решений линейного уравнения Осколкова в построенных пространствах в терминах устойчивого и неустойчивого инвариантных пространств и экспоненциальных дихотомий решений. Стохастическое линейное уравнение Осколкова рассматривается как частный случай стохастического линейного уравнения соболевского типа, где в качестве производной берется производная Нельсона–Гликлиха, а в качестве неизвестного выступает случайный процесс. При различных значения параметров, входящих в уравнение Осколкова, показано существование устойчивого и неустойчивого инвариантных пространств.

Ключевые слова: уравнения соболевского типа; дифференциальные формы; производная Нельсона–Гликлиха; инвариантные пространства.

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