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On Solving Optimization Problems with Inexact Data

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Abstract: There are considered two kinds of optimization problems with interval uncertainty. The first kind is interval linear programming (ILP), the second kind is finding equilibrium position for interval von Neumann's model (bilinear problem). Definitions of different types of solutions for both kinds of problems and methods for finding these solutions are given. These methods imply using matrices of upper and lower bounds of initial interval data and reducing interval optimization problems to exact (ordinary) linear programming problems.

Keywords: Operational research; industrial and applied mathematics for production; linear programming

1. INTRODUCTIONS

Methods for solving linear programming (LP) problems were greatly developed in the second half of 20th century due to the development of mathematical theory together with hardware and software [Danzig (1963)-Karmarkar (1984)] So scientists believed that solving LP problems was a key for effective economy. But in fact optimal plans obtained from LP were often ineffective (i.e. were not applicable). There were different factors that could lead to this ineffectiveness. One of them was inexactness of initial data.

For real-world linear economic models, numerical values of input matrices items are obtained using statistical data and expert estimates, therefore there can be an uncertainty, which is commonly interval. Using of average values may cause ineffectiveness of optimal solution, because uncertainty wasn't taken into account properly. Another approach is stochastic linear programming. This method requires that probability distributions for initial data are known while in practice this requirement does not hold in most cases.

The first section considers different issues connected with interval linear programming such as weak and strong solvability/feasibility, the range of optimum. It is shown that various programming problems connected with these issues can be effectively solved using parallel computations.

The second section consider multi-sector von Neumann' model. Finding equilibrium position for this model is bilinear optimization problem. Still we can use methodology described in the first section. The problem of finding equilibrium in von Neumann's model (A, B), when only interval values of the matrices elements in the von Neumann's model are given is considered in [Panyukov et al. (2008, Ufa), Panyukov et al. (2009)]. It is shown that in case of multiplicative uncertainty both primal and dual von Neumann's rays are obtained by exact von Neumann's model with matrices of interval centers, and interval of the Frobenius number for interval von Neumann's model is obtained by finding equilibrium for two exact von Neumann's models with matrices of interval upper and lower bounds. Definitions of weak solution and strong solution were introduced.

2. INTERVAL LINEAR PROGRAMMING PROBLEM

Let $\mathbf{A}$ designate an *interval matrix* with $n \times m$ size

$$\mathbf{A} = [\underline{\mathbf{A}}; \overline{\mathbf{A}}] = [\mathbf{A_c} - \Delta; \mathbf{A_c} + \Delta], \tag{1}$$

where $\underline{\mathbf{A}}$, $\overline{\mathbf{A}}$ are point matrices of of interval *lower* and *upper* bounds of $\mathbf{A}$ correspondingly; Δ is a matrix of with exact (point) entries,

$$\Delta = [\Delta_{ij}]_{i=1,\dots,n;j=1,\dots,m},\ \Delta_{ij} \geq 0;$$

$\mathbf{A_c}$ is a matrix of *interval centers* of $\mathbf{A}$ [Jaulin et al. (2001)], i.e.

$$\mathbf{A_c} = (\overline{\mathbf{A}} + \underline{\mathbf{A}})/2 = \underline{\mathbf{A}} + \Delta = \overline{\mathbf{A}} - \Delta. \tag{2}$$

Let us designate *interval vectors*

$$\mathbf{b} = [\underline{\mathbf{b}}; \overline{\mathbf{b}}] = [\mathbf{b}_c - \delta; \mathbf{b}_c + \delta]$$

and

$$\mathbf{c} = [\underline{c}; \overline{c}] = \mathbf{c}_c - \gamma; \mathbf{c}_c + \gamma].$$

Example. Let $\mathbf{A}$ be 2×2 matrix with lower and upper bounds

$$\underline{\mathbf{A}} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix},\ \overline{\mathbf{A}} = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}.$$

So corresponding matrices $\mathbf{A_c}$ and Δ are

$$\mathbf{A_c} = \begin{pmatrix} 2.5 & 0.5 \\ 1.5 & 2.5 \end{pmatrix},\ \Delta = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

Let us consider first the solvability (feasibility) problem of interval equations and inequalities.

A *weak solution* of system of interval linear equations

$$\mathbf{A}x = \mathbf{b} \tag{3}$$

is a vector $x \in \mathbf{R}^m$ which satisfies

$$Ax = b \tag{4}$$

for **some** matrices $A \in \mathbf{A}, b \in \mathbf{b}$. The following theorem is proved by W. Oettli and W. Pranger [Fiedler et al. (2006)].

Example. Let interval vector $\mathbf{b}$ has n entries

$$\underline{\mathbf{b}} = \begin{pmatrix} 0 \\ 60 \end{pmatrix},\ \overline{\mathbf{b}} = \begin{pmatrix} 120 \\ 240 \end{pmatrix},$$

$$\mathbf{b_c} = \begin{pmatrix} 60 \\ 150 \end{pmatrix},\ \delta = \begin{pmatrix} 120 \\ 180 \end{pmatrix}.$$

Theorem 1. The vector $x \in \mathbf{R}^m$ is a weak solution for system $\mathbf{A}x = \mathbf{b}$ if and only if x satisfies

$$|\mathbf{A}_c x - \mathbf{b}_c| \leq \Delta|x| + \delta. \tag{5}$$

It comes from theorem (1) that checking weak solvability of linear interval equations system (3) is NP-hard.

Example. For given in examples above matrix $\mathbf{A}$ and vector b system (5) will be

$$\left| \begin{pmatrix} 2.5 & 0.5 \\ 1.5 & 2.5 \end{pmatrix} x - \begin{pmatrix} 120 \\ 180 \end{pmatrix} \right| \leq \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} |x| + \begin{pmatrix} 60 \\ 150 \end{pmatrix}.$$

The system of linear interval equations (3) is *strongly solvable* if **every** system of exact linear equations $Ax = b$ is solvable, where $A \in \mathbf{A}, b \in \mathbf{b}$. Conditions under which system of interval linear equations in strongly solvable is presented in [Fiedler et al. (2006)]. Checking strong solvability of linear interval equations system (3) is NP-hard.

A *strong solution* of linear interval equations system (3) is a vector $x \in \mathbf{R}^m$ if it satisfies $Ax = b$ for **all** matrices $A \in \mathbf{A}, b \in \mathbf{b}$.

Theorem 2. [Fiedler et al. (2006)]
Vector $x \in \mathbf{R}^m$ is a strong solution of for system $\mathbf{A}x = \mathbf{b}$ if and only if x satisfies

$$\begin{gathered} \mathbf{A}_c x = \mathbf{b}_c; \\ \Delta|x| = \delta = 0. \end{gathered} \tag{6}$$

This theorem shows that strong solutions exist only in rare cases.

Let us consider solvability problem for systems of interval linear inequalities.

A *weak* solution of system of linear inequalities

$$\mathbf{A}x \leq \mathbf{b} \tag{7}$$

is a vector $x \in \mathbf{R}^m$ which satisfies

$$Ax \leq b \tag{8}$$

for **some** matrices $A \in \mathbf{A}, b \in \mathbf{b}$.

Theorem 3. [Fiedler et al. (2006)]
Vector $x \in \mathbf{R}^m$ is a weak solution of for system $\mathbf{A}x \leq \mathbf{b}$ if and only if x satisfies

$$\mathbf{A}_c x - \Delta|x| \leq \overline{\mathbf{b}}. \tag{9}$$

It comes from theorem (3) that checking weak solvability of linear interval inequalities system (7) is NP-hard.

Linear interval inequalities system (7) is *weakly feasible* if it has a nonnegative solution

$$Ax \leq b \tag{10}$$

for **some** matrices $A \in \mathbf{A}, b \in \mathbf{b}$.

Theorem 4. [Fiedler et al. (2006)]
System (7) is weakly feasible if and only if the system

$$\underline{\mathbf{A}}x \leq \overline{\mathbf{b}} \tag{11}$$

is feasible.

It comes from theorem (4) that checking weak feasibility of linear interval inequalities system (7) has polynomial complexity.

Example. For given in examples above matrix $\mathbf{A}$ and vector b system (7) will be

$$\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} x \leq \begin{pmatrix} 120 \\ 240 \end{pmatrix}.$$

Linear interval inequalities system (7) is *strongly solvable* if it has solution

$$Ax = b \tag{12}$$

for **all** matrices $A \in \mathbf{A}, b \in \mathbf{b}$.

Theorem 5. [Fiedler et al. (2006)]
System (7) is strongly solvable if and only if the following system (13) is feasible

$$\left\{ \begin{array}{c} \overline{\mathbf{A}}x^1 - \underline{\mathbf{A}}x^2 \leq \underline{\mathbf{b}}, \\ x^1 \geq 0, x^2 \geq 0, \end{array} \right\} \tag{13}$$

where two vectors $x^1, x^2 \in \mathbf{R}^m$ are solution of system (13).

Note that solving exact linear inequalities system (13) has polynomial complexity. So checking strong solvability for interval inequalities unlike interval equations needs polynomial time. This paradox comes from the fact that transition from equations to inequalities changes properties of system.

Example. For given in examples above matrix $\mathbf{A}$ and vector b system (13) will be

$$\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} x^1 - \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} x^2 \leq \begin{pmatrix} 0 \\ 60 \end{pmatrix}.$$

Linear interval inequalities system (7) has a *strong solution* x if vector x satisfies

$$Ax \leq b \tag{14}$$

for **all** matrices $A \in \mathbf{A}, b \in \mathbf{b}$.

It can be proved that if system (13) has feasible solution (x^1, x^2) then vector $x = x^1 - x^2$ is a strong solution for (7) [Fiedler et al. (2006)].

An interval linear programming problem (ILP problem) is a family of exact linear programming problems (LP problems)

$$\min\{c^T x |\, Ax = b,\, x \geq 0\}, \tag{15}$$

for which input matrices satisfy

$$A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}. \tag{16}$$

Example. Let interval vector $\mathbf{c}$ has n entries

$$\underline{\mathbf{c}} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}, \; \overline{\mathbf{c}} = \begin{pmatrix} 20 \\ 40 \end{pmatrix},$$

$$\mathbf{c_c} = \begin{pmatrix} 15 \\ 30 \end{pmatrix}, \; \gamma = \begin{pmatrix} 5 \\ 10 \end{pmatrix}.$$

Let $f(A, b, c)$ designate tne optimum for (15) and given matrices (A, b, c). *The range of optimum* for ILP (15-16) [Fiedler et al. (2006)] has the lower bound

$$\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \inf\{f(A, b, c) |\, A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\} \qquad (17)$$

and the upper bound

$$\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \sup\{f(A, b, c) |\, A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\}. \qquad (18)$$

Note that the interval

$$[\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}); \overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})] \qquad (19)$$

for this range may have infinite boundaries. Let us introduce the auxiliary nonlinear problem [Fiedler et al. (2006)]

$$\overline{\varphi} = \sup\{\mathbf{b}_c^T p + \delta^T |p| \,|\, \mathbf{A}_c^T p - \Delta^T |p| \le \overline{\mathbf{c}}\}. \qquad (20)$$

Example. For given matrices $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ problem (20) will be

$$\overline{\varphi} = \sup \left\{ \begin{matrix} \begin{pmatrix} 60 \\ 150 \end{pmatrix}^T p + \begin{pmatrix} 120 \\ 180 \end{pmatrix}^T |p| \,|\, \begin{pmatrix} 2.5 & 0.5 \\ 1.5 & 2.5 \end{pmatrix}^T p \\ - \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}^T |p| \le \begin{pmatrix} 20 \\ 40 \end{pmatrix} \end{matrix} \right\}.$$

Let $n \times 1$ vector y satisfy

$$y = \operatorname{sgn} p, \qquad (21)$$

i.e. there two alternatives for every element of vector y

$$y_i \in \{-1; 1\}, i = 1, 2, ..., n. \qquad (22)$$

So there are 2^n combinations for vector y, the set of these combinations can be denoted Y^n.

Example. If $n = 2$ then

$$Y^n = \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

If vector y is fixed then we can solve LP subproblem for (20)

$$\varphi(y) = \max \left\{ \mathbf{b}_c^T p + \delta^T (y^T p) \,|\, \begin{matrix} \mathbf{A}_c^T p - \Delta^T (y^T p) \le \overline{\mathbf{c}}, \\ (y^T p) \ge 0 \end{matrix} \right\}. \qquad (23)$$

Value of $\varphi(y)$ may be infinite. After using all combinations for vector y we can calculate upper bound $\overline{\varphi}$ this way

$$\overline{\varphi} = \sup\{\varphi(y) \,|\, y \in Y^n\}. \qquad (24)$$

Theorem 6. [Fiedler et al. (2006)]
For ILP problem $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ the following statements are equivalent:

(1) for any $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$ the problem

$$\min\{c^T x \,|\, Ax = b, \; x \ge 0\} \qquad (25)$$

has optimal solution;

(2) both lower bound $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ and upper bound $\overline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ of the range of optimum are finite;

(3) both lower bound $\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ and $\overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ are finite;

(4) system of inequalities

$$\overline{\mathbf{A}}^T p_1 - \underline{\mathbf{A}}^T p_2 \le \underline{\mathbf{c}} \qquad (26)$$

is feasible and $\overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is finite.

For every case (1)-(4) the range of optimum is equal to

$$[\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}); \overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c})] \qquad (27)$$

The proof of theorem (6) is given in [Fiedler et al. (2006)]. This book also contains the **algorithm** for calculating $[\underline{f}(\mathbf{A}, \mathbf{b}, \mathbf{c}); \overline{\varphi}(\mathbf{A}, \mathbf{b}, \mathbf{c})]$. This algorithm consists of two main steps:
Step 1. Calculation of $\underline{f}$ by solving a LP problem

$$\min\{\underline{\mathbf{c}}^T x \,|\, \underline{\mathbf{A}} x \le \overline{\mathbf{b}}, \overline{\mathbf{A}} x \ge \underline{\mathbf{b}}, \; x \ge 0\} \qquad (28)$$

Step 2. Calculation of $\overline{\varphi}$ by solving (20).

The first step requires solving the only LP problem that is why it has polynomial complexity [Fiedler et al. (2006)]. The second step requires solving of 2^n LP problems (23) therefore it has exponential complexity (in the worst case). Today LP problems can be effectively solved using parallel computations and exact rational-fractional calculations [Panyukov et al. (2012)]. It is evident that each subproblem (23) can be solved separately. So problem (20) has rather great potential for parallelism although coarse-grained parallelism is often used for each subproblem (23). CUDA C software engineering is suggested by author for parallel calculations.

Another approach to solve ILP problem (15) is interval simplex-method [Jaulin et al. (2001)]. The idea is to use rules of interval arithmetics to calculate elements of simplex-table. But this method has a lot of restrictions and disadvantages, so it is rather controversial.

Interval uncertainty leads to uncertainty of basic set of variables that is why it is difficult to estimate the set of optimal solutions. This set can be to wide to make effective solutions [Fiedler et al. (2006)].

3. EQUILIBRIUM POSITION OF INTERVAL VON NEUMANN'S MODEL

A *general* equilibrium position for von Neumann's model (A, B), where A and B are given $n \times m$ input and output matrices with numerical nonnegative items

$$a_{ij} \ge 0, \; b_{ij} \ge 0, \; i = 1, 2, \ldots n, \; j = 1, 2, \ldots m$$

is defined as a solution $(\lambda, \; x, \; w)$ of the system of bilinear inequalities and equations

$$(A - \lambda B)\, x \le 0, \; (x, e^m) = 1, \; x \ge 0, \qquad (29)$$

$$(A - \lambda B)^T \, w \ge 0, \; (w, e^n) = 1, \; w \ge 0, \qquad (30)$$

here $e^l \in \mathbf{R}^l$, $l \in \{m, n\}$ so that $(\forall i = 1, 2, \ldots, l) \; e_i^l = 1$.

A *nondegenerate* equilibrium position in the model under examination is an equilibrium position (λ, x, w) satisfying the additional condition (λ, x, w)

$$w^T A\, x > 0. \qquad (31)$$

In this paper, we consider algorithms for finding general equilibriums; i.e., we want to find a solution (λ, x, w) to system (29)-(30).

The extreme feasible values of λ can be found by solving the bilinear optimization problems

$$\lambda^* = \min\left\{\begin{array}{l} (A-\lambda B)\,x \le 0, \\ \lambda|\,(x, e^m) = 1, \\ x \ge 0 \end{array}\right\}, \tag{32}$$

$$\lambda_n = \max\left\{\begin{array}{l} (A-\lambda B)^T\,w \ge 0, \\ \lambda|\,(w, e^n) = 1, \\ w \ge 0 \end{array}\right\}. \tag{33}$$

The numbers λ_n and λ^* are called von Neumann and Frobenius numbers of the model (A, B) respectively. The von Neumann number λ_n determines the maximum possible balanced growth rate, while the Frobenius number λ^* determines the minimum possible balanced growth rate and the workability of the model (see [Ashamnov (1984), Al'sevich (2005)]). Vectors x, w of equilibrium position (λ, x, w) are called primal and dual von Neumann's rays corresponding the value of λ.

An *isolated pair* for von Neumann's model is a pair of arbitrary subsets $S \subset 1, 2, ..., m$ and $T \subset 1, 2, ..., n$, for which if $j \in S$ and $i \notin T$ then

$$a_{ij} = b_{ij} = 0.$$

If there is no isolated pair in von Neumann's model then the von Neumann's number and the Frobenius number coincide [Ashamnov (1984)].

Equalities (29), (30) and (32) imply that finding parameters of productivity (the Frobenius number λ^*) and stable equilibrium position for von Neumann's model lies in solving the following bilinear programming problem

$$(\lambda^*, x^*, w^*)^T = \arg \max_{(\lambda,x,w)^T \in D(A,B)} \lambda, \tag{34}$$

$$D(A,B) = \left\{ \begin{pmatrix} \lambda \\ x \\ w \end{pmatrix} \left| \begin{array}{c} (A-\lambda B)x \le 0, \\ (A-\lambda B)^T w \ge 0, \\ (x, e^m) = 1, \\ (w, e^n) = 1, \\ x \ge 0, w \ge 0, \lambda \ge 0 \end{array} \right. \right\}.$$

Hereinafter, parameters of productivity are meant by equilibrium position if it is not defined exactly.

Numerical methods of solving this problem (34) are discussed in [Panyukov et al. (2008, Moscow)]. They are based on finding the roots of the monotone function

$$u(\lambda) = \min_{x:\,(x,e^m)=1,\, x\ge 0} \; \max_{i=1,2,\ldots n} \sum_{j=1}^{m} (a_{ij} - \lambda b_{ij})\, x_j$$

or

$$v(\lambda) = \max_{w:\,(w,e^n)=1,\, w\ge 0} \; \min_{j=1,2,\ldots m} \sum_{i=1}^{n} (a_{ij} - \lambda b_{ij})\, w_i.$$

When λ is fixed, the values of the functions $u(\lambda)$ and $v(\lambda)$ equal to optimal values of mutually dual linear programming problems

$$\min\left\{ u \mid \begin{array}{l} (A-\lambda B)\,x \le u, \\ (x, e^m) = 1, \\ x \ge 0 \end{array}\right\}, \tag{35}$$

$$\max\left\{ v \mid; \mid \begin{array}{l} (A-\lambda B)^T\,w \ge v, \\ (w, e^n) = 1, \\ w \ge 0 \end{array}\right\}. \tag{36}$$

When λ is close to the roots of $u(\lambda)$, $v(\lambda) \;\to 0$ problems (35) and (36) become degenerate because of appearance of zero basic variables u and v in optimal basis solution of this problems. That is why problems (35) and (36) cannot be solved with conventional means based on floating-point arithmetic.

Let us present authors' results to solve the problem of finding the roots of functions $u(\lambda)$ and $v(\lambda)$ by numerical methods of matrix games. The first theorem can be used in case of absence of zero elements in matrix B.

Theorem 7. [Panyukov et al. (2008, Moscow)]
Let (A, B) be von Neumann's model, and all items of the matrix B are positive. Let Γ_C be the matrix game with payoff matrix $C = [a_{ij}/b_{ij}]$ $(i = 1, 2, \ldots, n,\ \ j = 1, 2, \ldots, m)$, and let $\hat{\lambda}$ be the value of the game Γ_C. Then the von Neumann number λ_n and the Frobenius number λ^* of this von Neumann's model are equal to $\hat{\lambda}$.

The second theorem is applicable if matrix A has no zero entries.

Theorem 8. [Panyukov et al. (2009)]
Let (A, B) be von Neumann's model, and all items of matrix A are positive. Let Γ_C be matrix game with payoff matrix $C = [b_{ij}/a_{ij}]$ $(i = 1, 2, \ldots, n,\ \ j = 1, 2, \ldots, m)$, and let $\hat{\mu}$ be the value of the game Γ_C. Then the von Neumann number λ_n and the Frobenius number λ_n of this von Neumann's model are equal to $1/\hat{\mu}$.

Note that theorem 7 (8) holds only for von Neumann's models in which the matrix B (A) consists of positive entries. The following approach can be used when some elements of the matrixes A and B are equal to zero. Hence it has a much wider field of application. Nevertheless Theorems 7 and 8 propose solving the only matrix game, while the method described below requires to solve several matrix games.

Theorem 9. [Panyukov et al. (2008, Moscow)] Let Γ be a matrix game with the payoff array $(A - \lambda B)^T$, and $\hat{x}$, $\hat{w}$ be the optimal mixed strategies of the first and the second players, respectively. Let $\hat{u}$ be the value of Γ. Then, $(\hat{u},\ \hat{x})$ and $(\hat{u},\ \hat{w})$ are optimal solutions to problems (35) and (36), respectively.

There are effective iterative algorithms of solving matrix games. Appliance of linear programming in solving of matrix games follows from that game $\overline{\mathbf{\Gamma}}$ with payoff array

$$\overline{(A-\lambda B)^T} = (A-\lambda B)^T + \gamma I$$

has the value $\overline{\hat{u}} = \hat{u} + \gamma$ and optimal solutions $\overline{\hat{x}} = \hat{x}$, $\overline{\hat{w}} = \hat{w}$, where I – $(n \times m)$-matrix, which all elements are equal to 1,

$$\gamma = \max\{a_{ij} - \lambda b_{ij} :\ i = 1, \ldots, m,\ j = 1, \ldots, n\},$$

yet corresponding linear programming problems to the game with payoff matrix $\overline{(A - \lambda B)^T}$ admit nondegenerate optimal solution.

Example. Let von Neumann's model be

$$A = \begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix},\ B = \begin{pmatrix} 5 & 2 \\ 4 & 4 \end{pmatrix}.$$

This model has equilibrium position

$$\left\{ \begin{array}{c} \lambda^* = 0.75, \\ x^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ w^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array} \right\}.$$

Henceforward $\mathbf{A}$ and $\mathbf{B}$ designate *interval* input and output matrices, i.e. items of $\mathbf{A}$ and $\mathbf{B}$ are numerical intervals.

Theorem 10. [Panyukov et al. (2012)]
Let $(\mathbf{A}, \mathbf{B})$ be interval von Neumann's model, and the point matrices $\tilde{A}, \tilde{B}$ satisfy the condition

$$\tilde{A} \in \mathbf{A}, \quad \tilde{B} \in \mathbf{B}. \tag{37}$$

Let it is hold

$$(\tilde{\lambda}, \tilde{x}, \tilde{w})^T = \arg \max_{(\lambda,x,w)^T \in D(\tilde{A},\tilde{B})} \lambda\,; \tag{38}$$

$$(\overline{\lambda}, \overline{x}, \overline{w})^T = \arg \max_{(\lambda,x,w)^T \in D(\overline{\mathbf{A}},\underline{\mathbf{B}})} \lambda\,; \tag{39}$$

$$(\underline{\lambda}, \underline{x}, \underline{w})^T = \arg \max_{(\lambda,x,w)^T \in D(\underline{\mathbf{A}},\overline{\mathbf{B}})} \lambda. \tag{40}$$

Then $\underline{\lambda} \le \tilde{\lambda} \le \overline{\lambda}$.

By theorem 10, one might that interval von Neumann's model $(\mathbf{A}, \mathbf{B})$ has Frobenius interval $\mathbf{\Lambda} = [\underline{\lambda}, \overline{\lambda}]$, where $\overline{\lambda}$ is the Frobenius number of exact von Neumann's model $(\overline{\mathbf{A}}, \underline{\mathbf{B}})$, and $\underline{\lambda}$ is the Frobenius number of exact von Neumann's model $(\underline{\mathbf{A}}, \overline{\mathbf{B}})$. Under that any exact von Neumann's model $(\tilde{A}, \tilde{B}) \in (\mathbf{A}, \mathbf{B}) : \ \tilde{A} \in \mathbf{A},\ \tilde{B} \in \mathbf{B}$ has the Frobenius number $\tilde{\lambda} \in \mathbf{\Lambda}$.

Example. Let interval von Neumann's model be

$$\underline{\mathbf{A}} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}, \overline{\mathbf{A}} = \begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix},$$

$$\underline{\mathbf{B}} = \begin{pmatrix} 5 & 2 \\ 4 & 4 \end{pmatrix},\ \overline{\mathbf{B}} = \begin{pmatrix} 6 & 3 \\ 4 & 5 \end{pmatrix}.$$

This model has range for the Frobenius number [0.4, 0.75].

Theorem 11. [Panyukov et al. (2012)]
Let $(\mathbf{A}, \mathbf{B})$ be interval von Neumann's model, and numbers $\beta_A,\ \beta_B$ and matrices $\tilde{A},\ \tilde{B}$ satisfy the conditions

$$\tilde{A} = \beta_A \cdot \mathbf{A}_c \in \mathbf{A},\ \tilde{B} = \beta_B \cdot \mathbf{B}_c \in \mathbf{B}.$$

Then equality

$$(\lambda^*, x^*, w^*)^T = \arg \max_{(\lambda,x,w)^T \in D(\mathbf{A}_c,\mathbf{B}_c)} \lambda$$

implies

$$\left(\frac{\lambda^* \beta_A}{\beta_B},\ x^*,\ w^* \right)^T = \arg \max_{(\lambda,x,w)^T \in D(\tilde{A},\tilde{B})} \lambda\,.$$

Thus, in case of multiplicative uncertainty although the Frobenius number changes, both primal and dual Frobenius vectors stay the same. But such stability can be observed in case of more general interval uncertainty.

Strong solution (x_s, w_s) of interval model $(\mathbf{A}, \mathbf{B})$ provides equilibrium position $(\tilde{\lambda}, x_s, w_s)$ for any exact von Neumann's model $(\tilde{A}, \tilde{B}) : \quad \tilde{A} \in \mathbf{A},\ \tilde{B} \in \mathbf{B}$.

Example. The model (A, B) with 2×2-size and elements, which satisfy the given constraints

$$\left\{ \begin{array}{l} a_{21} + a_{22} \le \lambda^*(b_{21} + b_{22}); \\ a_{11} = b_{11}; \\ a_{12} = b_{12}; \end{array} \right\}$$

has the equilibrium position

$$\left\{ \begin{array}{l} \lambda^*, (x^*)^T = (0.5; 0.5), \\ ; (w^*)^T = (1; 0). \end{array} \right\}$$

Existence of weak solution provides feasibility of set of constraints (29)-(30) for interval model $(\mathbf{A}, \mathbf{B})$.

Weak solution for interval von Neumann's model $(\mathbf{A}, \mathbf{B})$ is a pair of vectors (x', w'), under which set of constraints

$$\left\{ \begin{array}{l} (\tilde{A} - \lambda \tilde{B}) x' \le 0; \\ (\tilde{A} - \lambda \tilde{B})^T w' \ge 0; \\ (x', e^m) = 1; \\ (w', e^n) = 1; \\ x', w', \lambda \ge 0. \end{array} \right.$$

is feasible for any exact von Neumann's model $(\tilde{A}, \tilde{B})$: $\tilde{A} \in \mathbf{A},\ \tilde{B} \in \mathbf{B}$.

Note if pair of vectors (x_s, w_s) is a strong solution of interval model $(\mathbf{A}, \mathbf{B})$ then (x_s, w_s) is a weak solution for the given model.

Proposition 12. Interval von Neumann's model $(\mathbf{A}, \mathbf{B})$ has a strong solution only if $\underline{\lambda} = \overline{\lambda}$.

If some entries of matrix A or B are changed then the set of non-zero variables for vectors x and w may change.

The following statement can be used to define whether pair of vectors(x, w) is a weak solution.

Theorem 13. [Panyukov et al. (2012)]
If set of constraints

$$\left\{ \begin{array}{l} (\overline{\mathbf{A}} - \underline{\lambda}\, \underline{\mathbf{B}}) x'' \le 0; \\ (\underline{\mathbf{A}} - \overline{\lambda}\, \overline{\mathbf{B}})^T w'' \ge 0; \\ (x'', e^m) = 1; \\ (w'', e^n) = 1; \\ x'', w'' > 0; \end{array} \right.$$

is feasible under pair of vectors (x'', w'') then (x'', w'') is a weak solution of interval von Neumann's model $(\mathbf{A}, \mathbf{B})$.

According to the theorem (13) if $\underline{\lambda} = \overline{\lambda}$ then pair of vectors $(\overline{x}, \underline{w})$ is a weak solution of interval von Neumann's model $(\mathbf{A}, \mathbf{B})$.

If model $(\mathbf{A}, \mathbf{B})$ has a weak solution, this solution can be used for estimating of the Frobenius number by λ'.

Theorem 14. [Panyukov et al. (2012)]
Let $(x',\ w')$ be a weak solution for interval von Neumann's model $(\mathbf{A}, \mathbf{B})$. If exact von Neumann's model $(\tilde{A}, \tilde{B})$: $\tilde{A} \in \mathbf{A},\ \tilde{B} \in \mathbf{B}$ has

$$\lambda' = \max \left\{ \lambda | \begin{array}{l} (\tilde{A} - \lambda \tilde{B}) x' \le 0; \\ (\tilde{A} - \lambda \tilde{B})^T w' \ge 0 \end{array} \right\},$$

then $\lambda' \in [\underline{\lambda}_n; \overline{\lambda}]$, where $\underline{\lambda}_n$ is the von Neumann's number for model $(\underline{\mathbf{A}}, \overline{\mathbf{B}})$.

According to theorem (14) estimate λ' of the Frobenius number may be less then $\underline{\lambda}$. If exact von Neumann's model $(\underline{\mathbf{A}}, \overline{\mathbf{B}})$ has no isolated pairs then $\underline{\lambda}_n = \underline{\lambda}$.

The following theorem suggests another approach for estimating the Frobenius number.

Theorem 15. [Panyukov et al. (2012)]
Let (x^*, w^*) be primal and dual Frobenius vectors for two exact von Neumann's models $(\breve{A}, \breve{B})$ and $(\hat{A}, \hat{B})$, for which

$$\hat{a}_{ij} - \breve{a}_{ij} = \Delta a_{ij} \geq 0,$$
$$\hat{b}_{ij} - \breve{b}_{ij} = \Delta b_{ij} \geq 0.$$

Let the Frobenius number of model $(\breve{A}, \breve{B})$ be equal to $\breve{\lambda}$, and $(\hat{A}, \hat{B})$ has the Frobenius number $\hat{\lambda}$,

$$\Delta\lambda = \hat{\lambda} - \breve{\lambda}.$$

Then if equilibrium position $(\breve{\lambda}, x^*, w^*)$ is not degenerate for model $(\breve{A}, \breve{B})$,

$$\Delta\lambda = \frac{(w^*)^T (\Delta_A - \breve{\lambda}\Delta_B) x^*}{(w^*)^T (\breve{B} + \Delta_B) x^*}. \quad (41)$$

If interval von Neumann's model $(\mathbf{A}, \mathbf{B})$ has a strong solution (x_s, w_s), we can estimate change of the Frobenius number for exact von Neumann's models.

Theorem 16. [Panyukov et al. (2012)]
Let (x_s, w_s) be a strong solution for interval von Neumann's model $\left([\breve{A}, \hat{A}], [\breve{B}, \hat{B}]\right)$, both von Neumann's models $(\breve{A}, \breve{B})$ and $(\hat{A}, \hat{B})$ have nondegenerate equilibrium positions $(\breve{\lambda}, x_s, w_s)$ and $(\hat{\lambda}, x_s, w_s)$ respectively. Let exact von Neumann's model $(\tilde{A}, \tilde{B})$: $\tilde{A} = \breve{A} + k\Delta_A$, $\tilde{B} = \breve{B} + k\Delta_B$, $k \in [0; 1]$ has the Frobenius number $\tilde{\lambda}$. Then

$$\tilde{\lambda} - \breve{\lambda} = k\Delta\lambda \frac{w_s^T (\breve{B} + \Delta_B) x_s}{w_s^T (\breve{B} + k\Delta_B) x_s}. \quad (42)$$

It comes from (42) that changing of the Frobenius number can be nonuniform.

4. CONCLUSIONS

This article concerns optimization problems with inexact data. One of such problems is interval linear programming. The main approach described is usage of lower and upper bounds to find different types of solutions. These solutions can be obtained by solving series of exact LP problems with using of parallel computations. It was shown that bilinear problem of finding equilibrium position for interval von Neumann's model can be treated similarly.

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