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Estimation of accuracy of finite-dimensional methods of regularization

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Abstract. We consider the estimation of the accuracy of finite dimensional approximations of regularized solutions for inverse problems in a separable Hilbert space. A connection of the regularization parameter with the estimates of error for these approximations and with the estimates of initial date is obtained. For the establishment method the estimation of error of the finite-dimensional approximation is obtained. This estimate is exact with respect to the order.

Keywords. Operator equation, correctness class, finite-dimensional methods, estimation of accuracy of method, establishment method.

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1 Introduction

Research of ill-posed problems is impossible without using calculating methods. Therefore the projection finite difference and other discretizations which reduce an initiate problem to a finite-dimensional one [3, 7, 10] play an important role. Until now the ill-posed problems were solved in two stages. First we choose the optimal method and find the error estimate for it. Then, using the discretization scheme we construct the finite-dimensional variant of this method. However, researching finite-dimensional approximations we restrict ourselves only to the problem on convergence of approximations. Such approach of using finite-dimensional approximation had led to uncontrollable estimates. As a result the high accuracy of the method got lost.

In our paper we use the approach suggested in [8], which connected the regularization parameter both with the error of initial date and with the error of finite-dimensional approximation of the initial problem. This allows us to consider the finite-dimensional regularization methods of solution of the basic problem and to obtain the estimates exact with respect to the estimate order. This approach is illustrated by the example of the establishment method [6].

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2 Setting of the problem

Let U, F and V be reflexive Banach spaces; $(U \to F)$ be the space of linear bounded operators mapping U into F; $(U \to F)_1$ be the set of linear injective completely continuous operators $A : U \to F$, whose set of values $R(A)$ is the set everywhere dense in F; $(U \to F)_2$ be the linear manifold of linear finite-dimensional operators mapping U into F; $B : V \to U$ be a linear completely continuous operator; and $M_r = B\overline{S_r}$ where $\overline{S}_r = \{v : v \in V, \|v\| \le r\}$.

Consider the operator equation of the first kind

$$Au = f, \tag{2.1}$$

where $u \in M_r$, $f \in F$, $A \in (U \to F)_1$.

We assume that for a certain f_0 there exists a precise solution $u_0 \in M_r$ of equation (2.1), but instead of f we know its approximations $f_\delta \in F$, and the estimate level $\delta > 0$ such that $\|A_h - A\| \le h$, $\|f_\delta - f\| \le \delta$.

Consider now the set of operators $\{P_\alpha(A) : 0 < \alpha \le \alpha_0\}$ strongly continuous with respect to a mapping $(U \to F) \times F$ into U and such that for all $\alpha \in (0; \alpha_0]$ and for all $A \in (U \to F)$ we have $P_\alpha(A) \in (F \to U)$. Then following [1], we shall call the set of operators $\{P_\alpha(A) : 0 < \alpha \le \alpha_0\}$ by the regularizing operator set for equation (2.1) in the space U, if for each injective operator $A \in (U \to F)$ and for all $u \in U$ the following relation holds:

$$P_\alpha(A)Au \longrightarrow u \quad \text{for} \ \alpha \to 0.$$

Now we reduce equation (2.1) to the finite-dimensional one. To this end, we consider the operator $A_h \in (U \to F)_2$ such that $\|A_h - A\| \le h$, $h > 0$, and the operator of metric projection $\mathrm{pr}[\cdot; R(A_h)]$ of the space F on the set of values $R(A_h)$ of the operator A_h. In order that the operator $\mathrm{pr}[\cdot; R(A_h)]$ be one-to-one and continuous if suffices that the space F be strongly convex. Then the finite-dimensional equation

$$A_h u = \overline{f_\delta} \tag{2.2}$$

will correspond to equation (2.1) where $\overline{f_\delta} = \mathrm{pr}[f_\delta; R(A_h)]$, $\|f_\delta - f_0\| \le \delta$ and the values h and δ are known.

Apply to equation (2.2) regularizing set of operators $\{P_\alpha(A) : 0 < \alpha \le \alpha_0\}$ and call the element $u^\alpha_{\delta h} = P_\alpha(A_h)\overline{f_\delta}$ by the regularized solution of equation (2.1) with the approximate initial data A_h and f_δ.

In order to estimate the deviation of the regularized solution $u^\alpha_{\delta h}$ from the exact solution u_0 in the class M_r, we consider the following function [8]:

$$\Delta[P_\alpha(A)] = \sup_{u, f_\delta, A_h} \big\{\|P_\alpha(A_h)\,\mathrm{pr}[f_\delta; R(A_h)] - u\| : \\ u \in M_r,\ \|A_h - A\| \le h, \|Au - f_\delta\| \le \delta\big\}. \quad (2.3)$$

3 Definition and the properties of the continuity modules of the inverse operator

Let the injective operator $A \in (U \to F)$. Then following [2], define the functions $\omega_1(\tau, r)$ and $\omega(\tau, r)$ as follows:

$$\omega_1(\tau, r) = \sup\big\{\|u_1 - u_2\| : u_1,\ u_2 \in M_r,\ \|Au_1 - Au_2\| \le \tau\big\}, \quad (3.1)$$

$$\omega(\tau, r) = \sup\big\{\|u\| : u \in M_r,\ \|Au\| \le \tau\big\}. \quad (3.2)$$

The following statement [7] holds.

Lemma 1. *Let the functions $\omega_1(\tau, r)$ and $\omega(\tau, r)$ be defined by formulas* (3.1) *and* (3.2)*. Then*

$$\omega_1(\tau, r) = \omega(\tau, 2r).$$

Lemma 2. *Let $k \ge 0$. Then*

$$\omega(k\tau, kr) = k\omega(\tau, r).$$

Proof. The lemma evidently holds if $k = 0$. Let $k > 0$ and $\tau \ge r\|AB\|$. Then $k\tau \ge kr\|AB\|$ and (3.2) yields

$$\omega(\tau, r) = r\|AB\| \quad (3.3)$$

and

$$\omega(k\tau, kr) = kr\|AB\|. \quad (3.4)$$

From (3.3) and (3.4) it follows that $\omega(k\tau, kr) = k\omega(\tau, r)$.

Let $k > 0$ and $\tau < r\|AB\|$. Then $ku \in M_{kr}$ and $\|A(ku)\| \le k\tau$. Thus

$$k\omega(\tau, r) \le \omega(k\tau, kr). \quad (3.5)$$

Conversely let $u \in M_{kr}$ and $\|Au\| \le k\tau$. Then $\frac{u}{k} \in M_r$ and $\|A(\frac{u}{k})\| \le \tau$, Therefore,

$$\frac{1}{k}\omega(k\tau, kr) \le \omega(\tau, r)$$

or

$$\omega(k\tau, kr) \le k\omega(\tau, r). \quad (3.6)$$

From (3.5) and (3.6) the statement of the lemma follows. □

Lemma 3. *The function $\omega(\tau, r) \in C([0;\infty) \times [0;\infty))$ is nondecreasing with respect to τ and r.*

Proof. Nondecrease of the function $\omega(\tau, r)$ with respect to τ and r follows from (3.2). Next, we show that the function is continuous. The following cases are possible. In the case when $\tau, r > 0$ we assume that $\tau_n \to \tau$, $r_n \to r$ for $n \to \infty$. Introduce the numbers

$$c_n = \frac{\tau + |\tau_n - \tau|}{\tau} \quad \text{and} \quad c'_n = \frac{\tau - |\tau_n - \tau|}{\tau}, \qquad \tau > 0, \tag{3.7}$$

$$d_n = \frac{r + |r_n - r|}{r} \quad \text{and} \quad d'_n = \frac{r - |r_n - r|}{r}, \qquad r > 0. \tag{3.8}$$

Denote $k_n = \max(c_n, d_n)$ and $k'_n = \min(c'_n, d'_n)$. Then from Lemma 2 and taking into account (3.2), (3.7), (3.8) we have

$$k'_n \omega(\tau, r) \le \omega(\tau_n, r_n) \le k_n \omega(\tau, r). \tag{3.9}$$

As $\lim_{n\to\infty} k_n = \lim_{n\to\infty} k'_n = 1$, the lemma statement follows for $\tau, r > 0$ from (3.9).

When $r = 0$, it follows from (3.2) that

$$\omega(\tau, r) = 0. \tag{3.10}$$

We now consider the remaining cases. Let $r_n \to 0$, $\tau_n \to \tau \ge 0$. Then taking (3.2) into account we obtain

$$\omega(\tau_n, r_n) \le r_n \|B\|, \tag{3.11}$$

and this yields $\omega(\tau_n, r_n) \to 0$ for $n \to \infty$.

Now, let $\tau_n \to 0$, $r_n \to r \ge 0$. Then there exists an $\overline{r} > 0$ such that we have $r_n \le \overline{r}$ for each n. The set $\overline{M}_n$ will be defined as follows:

$$\overline{M}_n = \{u : u \in B\overline{S}_{\overline{r}},\ \|Au\| \le \tau_n\}, \tag{3.12}$$

where $\overline{S}_{\overline{r}} = \{v :\ v \in V, \|v\| \le \overline{r}\}$. Since the set $\overline{M}_n$ defined in (3.12) is compact for all n, here exist an element $\overline{u}_n \in \overline{M}_n$ such that

$$\|\overline{u}_n\| = \sup\{\|u\| : u \in \overline{M}_n\}.$$

We show that $n \to \infty$ for $\overline{u}_n \to 0$. Assume the contrary: there exists $d > 0$ and the subsequence $\{\overline{u}_{n_k}\}$ such that for all k

$$\|\overline{u}_{n_k}\| \ge d. \tag{3.13}$$

As the subsequence $\{\overline{u}_{n_k}\}$ belongs to compact $B\overline{S}_{\overline{r}}$ there exists a subsequence $\overline{u}_{n_{k_l}}$ such that

$$\overline{u}_{n_{k_l}} \to \hat{u} \quad \text{for } l \to \infty. \tag{3.14}$$

From (3.13) and (3.14) it follows that

$$\hat{u} \neq 0 \tag{3.15}$$

and $A\overline{u}_{n_{k_l}} \to 0$ yields

$$A\hat{u} = 0. \tag{3.16}$$

The injective property of the operator A follows from (3.15) and (3.16). This contradicts the initial assumption and proves the lemma. □

4 Estimate of the error of the regularized solution in the case if the operator is given exactly

To estimate the error of the regularized solution when the operator is given exactly, we use the function $\Delta_0[P_\alpha(A)]$ defined by relation

$$\Delta_0[P_\alpha(A)] = \sup_{u, f_\tau} \big\{ \|P_\alpha(A) f_\tau - u\| : u \in M_r,\ \|Au - f_\tau\| \leq \tau \big\}. \tag{4.1}$$

The following estimate holds [3].

Lemma 4. *The function $\Delta_0\,[P_\alpha(A)]$ satisfies the inequality*

$$\frac{1}{2}\big\{\|[E - P_\alpha(A)A]B\| \cdot r + \|P_\alpha(A)\|\tau\big\} \leq \Delta_0\,[P_\alpha(A)] \\ \leq \|[E - P_\alpha(A)A]B\| \cdot r + \|P_\alpha(A)\|\tau.$$

Lemma 5. *If $\tau < r\|AB\|$, then $\Delta_0[P_\alpha(A)] \geq \omega(\tau, r)$.*

Proof. Since the set M_r is a compact set, there exist points $u_1, u_2 \in M_r$ so that $\|u_1 - u_2\| = \omega_1(2\tau, r)$ and $\|Au_1 - Au_2\| \leq 2\tau$. From Lemma 1 it follows that

$$\|u_1 - u_2\| = \omega(2\tau, 2r).$$

Hence and from Lemma 2, we have

$$\|u_1 - u_2\| = 2\omega(\tau, r). \tag{4.2}$$

Let $\tilde{f}_\tau = \frac{1}{2}(Au_1 + Au_2)$. Then $\|Au_1 - \tilde{f}_\tau\| \le \tau$ and $\|Au_2 - \tilde{f}_\tau\| \le \tau$, and for $f = \tilde{f}_\tau$, $\tilde{u}_\tau^\alpha = P_\alpha(A)\tilde{f}_\tau$ the following relation will hold:

$$\sup\{\|\tilde{u}_\tau^\alpha - u_1\|,\ \|\tilde{u}_\tau^\alpha - u_2\|\} \ge \frac{1}{2}\|u_1 - u_2\|. \tag{4.3}$$

The relations (4.1), (4.2) and (4.3) prove the lemma. □

5 The below estimate for the error $\Delta[P_\alpha(A)]$

Let $U = F = V = H$, where H be a separable Hilbert space and an operator $A \in (H \to H)_1$ be positive semidefinite, self-adjoint, completely continuous. Assume the set of strongly continuous operators with respect to α $\{P_\alpha(A) : 0 < \alpha \le \alpha_0\}$, and the regularizing equation (2.1) in the space H defined by the formula

$$P_\alpha(A) = \Phi(A, \alpha)Q, \quad 0 < \alpha \le \alpha_0. \tag{5.1}$$

Here $Q = \mathrm{pr}[\cdot, \overline{R(A)}]$, $\overline{R(A)}$ is the closure of values of the operator A, and $\Phi(\sigma, \alpha)$ is a nonnegative function continuous with respect to σ.

Assume that the function $\|P_\alpha(A)\|^{-1}$ does not decrease by α, and the operator B, generating the set M_r, is such that $B = g(A)$, where $g(\sigma)$ strongly increases and $\lim_{\sigma\to 0} g(\sigma) = 0$.

Lemma 6. *The function $\Phi(\sigma, \alpha)$ satisfies the relation*

$$\sup_{\sigma\in Sp(A)} \Phi(\sigma, \alpha) \longrightarrow \infty \quad \text{for } \alpha \to 0. \tag{5.2}$$

Proof. Assume the contrary, i.e. (5.2) fails. Then there exist K and the sequence $\{\alpha_n\}$ such that $\alpha_n \to +0$ for $n \to \infty$ and

$$\sup_{\sigma\in Sp(A)} \Phi(\sigma, \alpha_n) \le K \tag{5.3}$$

for each n.

Since $\|P_\alpha(A)\| \le \sup_{\sigma\in Sp(A)} \Phi(\sigma, \alpha)$ and from (5.3) it follows that for each n

$$\|P_\alpha(A)\| \le K. \tag{5.4}$$

The set of the values $R(A)$ and $R(B)$ of the operators A and B are dense in H; therefore the linear span $L[A(M_r)]$ of the set $A(M_r)$ is dense in H, i.e.

$$\overline{L[A(M_r)]} = H. \tag{5.5}$$

The properties (5.4), (5.5) and the fact that the set of operators $\{P_\alpha(A) : 0 < \alpha \le \alpha_0\}$ regularizes equation (2.1) in the set M_r, yield boundedness of the inverse operator A^{-1}. This contradicts the complete continuity of the operator A. The lemma is proved. □

In order to evaluate $\Delta[P_\alpha(A)]$ we consider the function

$$G(\alpha) = \sup_{\sigma \in Sp(A)} |(1 - \sigma\Phi(\sigma,\alpha))\, g(\sigma)|. \tag{5.6}$$

Lemma 7. *The function $G(\alpha)$ is continuous for $\alpha > 0$.*

Proof. Let $\alpha_0 > 0$ and $\alpha_n \to \alpha_0$ for $n \to \infty$, where $\alpha_n \subset R^+$. From the strong continuity of the operator set $\{P_\alpha(A) : 0 < \alpha \le \alpha_0\}$ it follows that for all $u_0 \in H$

$$P_{\alpha_n}(A)Au_0 \longrightarrow P_{\alpha_0}(A)Au_0 \quad \text{for } n \to \infty. \tag{5.7}$$

As $M_r = B\overline{S}_r$ is compact, uniform convergence of the operator sequence $P_{\alpha_n}(A)$ to the operator $P_{\alpha_0}(A)$ in the set $A(M_r)$ follows from (5.7).

Thus

$$\|P_{\alpha_n}(A)AB - P_{\alpha_0}(A)AB\| \longrightarrow 0 \quad \text{for } n \to \infty. \tag{5.8}$$

Hence it follows that

$$\Big|\|B - P_{\alpha_n}(A)AB\| - \|B - P_{\alpha_0}(A)AB\|\Big| \longrightarrow 0 \quad \text{for } n \to \infty. \tag{5.9}$$

From the forms of operators A, B and $P_\alpha(A)$ it follows that

$$\|B - P_\alpha(A)AB\| = \sup_{\sigma \in Sp(A)} |(1 - \sigma\Phi(\sigma,\alpha))\, g(\sigma)|. \tag{5.10}$$

Relations (5.6), (5.9) and (5.10) proof the lemma. □

Define the parameter α_τ as

$$\alpha_\tau = \sup\Big\{\alpha : G(\alpha) \le \tau \sup_{\sigma \in Sp(A)} \Phi(\sigma,\alpha)\Big\}. \tag{5.11}$$

Lemma 8. *Let the function $G(\alpha)$ be defined by relation* (5.6)*, and α_τ by* (5.11)*, where $\tau > 0$. Then*

$$\lim_{\tau \to 0} \alpha_\tau = 0.$$

Proof. Assume the contrary. Then there exists the value $d > 0$ and the sequence $\{\tau_n\}$ such that $\lim_{n\to\infty} \tau_n = 0$ and for all n

$$\alpha_{\tau_n} \geq d.$$

Let $\overline{\alpha}_n$ be the value of parameter α, such that

$$\overline{\alpha}_n \geq \frac{d}{2}. \tag{5.12}$$

and

$$G(\overline{\alpha}_n) \leq \| P_{\overline{\alpha}_n}(A) \| \tau_n. \tag{5.13}$$

Since $\| P_\alpha(A) \|^{-1}$ does not decrease by α and from (5.12), it follows that for each n

$$\| P_{\overline{\alpha}_n}(A) \| \leq \| P_{\frac{d}{2}}(A) \|. \tag{5.14}$$

Relations (5.13) and (5.14) yield

$$G(\overline{\alpha}_n) \longrightarrow 0 \quad \text{for} \;\; n \to \infty. \tag{5.15}$$

Select the converging subsequence $\{\overline{\alpha}_{n_k}\}$ from $\{\overline{\alpha}_n\}$ so that

$$\lim_{k\to\infty} \overline{\alpha}_{n_k} = \hat{\alpha} \geq \frac{d}{2}. \tag{5.16}$$

Continuity of the function $G(\alpha)$ installed in Lemma 7 and relation (5.15) yield

$$G(\hat{\alpha}) = 0. \tag{5.17}$$

From (5.10), (5.16) and (5.17) it follows that for each $f \in A(M_r)$

$$P_{\hat{\alpha}}(A) f = A^{-1} f.$$

Taking into account the boundedness of the operator $P_{\hat{\alpha}}(A)$ and the density of the linear span $L[A(M_r)]$ of the set $A(M_r)$, we obtain $P_{\hat{\alpha}}(A) f = A^{-1} f$ for each $f \in H$. This in turn implies boundedness of the operator A^{-1} and contradicts the initial assumption. □

We denote by σ_n the eigenvalues of the operator A, corresponding to its eigenelements e_n. The sequence σ_n is supposed to be nonincreasing and $\lim_{n\to\infty} \sigma_n = 0$. As the operator $\Phi(A, \alpha)$ is a function depending on the operator A, it has similar eigenelements e_n. Denote by $\lambda_n(\alpha)$ the corresponding eigenvalues of the operator $\Phi(A, \alpha)$.

From the properties of the operator norm it follows that there exists an eigen-element $e_{i(\alpha)}$ such that

$$\lambda_{i(\alpha)} = \lambda_{i(\alpha)}(\alpha) = \|P_\alpha e_{i(\alpha)}\| \geq \frac{\|P_\alpha(A)\|}{2}. \tag{5.18}$$

Let

$$\lambda_1(\alpha) = \|P_\alpha(A)e_1\|. \tag{5.19}$$

Lemma 9. *Suppose that $\lambda_1(\alpha)$ and $\lambda_{i(\alpha)}(\alpha)$ are defined by relations* (5.19) *and* (5.18). *Then there exists $\tilde{\alpha}_0$ such that for all $\alpha \leq \tilde{\alpha}_0$ the following relation holds:*

$$\lambda_1(\alpha) \leq \frac{\lambda_{i(\alpha)}(\alpha)}{9}.$$

Proof. Since $\alpha \to 0$, we have $P_\alpha(A)e_1 \longrightarrow A^{-1}e_1$. Then for $\alpha \to 0$ we obtain

$$\lambda_1(\alpha) \longrightarrow \|A^{-1}e_1\| = \sigma_1^{-1}, \tag{5.20}$$

where $\sigma_1^{-1} > 0$. From Lemma 6 it follows that

$$\|P_\alpha(A)\| \longrightarrow \infty \quad \text{for } \alpha \to 0. \tag{5.21}$$

On the basis (5.18), (5.21) we have

$$\lambda_{i(\alpha)}(\alpha) \longrightarrow \infty \quad \text{for } \alpha \to 0. \tag{5.22}$$

The statement of lemma follows from (5.20) and (5.22). □

We define now the number τ_0. For $0 < \delta \leq \delta_0$ and $0 < h \leq h_0$, where $h_0 < 2\|A\|$, let the number τ_0 satisfy the conditions

$$\tau_0 = r\|B\|h_0 + \delta_0 \tag{5.23}$$

and

$$\tau_0 < \|f_\delta\|. \tag{5.24}$$

The following statement holds.

Theorem 1. *Let $A \in (H \to H)_1$ be a nonnegative self-adjoint operator, the regularizing set $\{P_\alpha(A) : 0 < \alpha \leq \alpha_0\}$ be defined by relation* (5.1), *the regularization parameter α_{τ_0} by relation* (5.11), *where the number τ_0 satisfies the conditions* (5.23), (5.24), *and the eigenvalues $\lambda_1(\alpha)$, $\lambda_{i(\alpha)}(\alpha)$ be defined by relations* (5.19), (5.18). *Then the value $\Delta[P_\alpha(A)]$ satisfies the estimate*

$$\Delta[P_\alpha(A)] \geq \frac{1}{38}\omega(r\|B\|h + \delta, r).$$

Proof. The existence of the values δ_0 and h_0 follows from Lemma 8 and 9.

Further, we consider the two cases.

First case. Let the function $G(\alpha)$, given by (5.6), satisfy inequality

$$G(\alpha) \geq \frac{1}{18}\|P_\alpha(A)\|(r\|B\|h + \delta). \tag{5.25}$$

Denote by H_k the subspace of H, generated by the eigenelements $e_1, e_2, \ldots, e_k$ of the operator A. We choose the number k from the conditions

$$\sigma_{k+1} \leq h \tag{5.26}$$

and

$$G_k(\alpha) \geq \frac{1}{2}G(\alpha), \tag{5.27}$$

where

$$G_k(\alpha) = \sup_{u_0}\bigl\{\|u_0 - P_\alpha(A)Au_0\| : u_0 \in M_r \cap H_k\bigr\}. \tag{5.28}$$

As the operator A_h we take the contraction of the operator A in the subspace H_k, and the element f_δ we define by the relation $f_\delta = Au_0$, where $u_0 \in M_r \cap H_k$.

Thus, (5.26) yields

$$\|A - A_h\| \leq h \tag{5.29}$$

and

$$\|f_\delta - Au_0\| \leq \delta. \tag{5.30}$$

Taking into account (2.3) and (5.27)–(5.30) we obtain

$$\Delta[P_\alpha(A)] \geq \frac{1}{2}G(\alpha). \tag{5.31}$$

Relations (5.25) and (5.31) yield

$$\Delta[P_\alpha(A)] \geq \frac{1}{38}\bigl\{G(\alpha) + \|P_\alpha(A)\|(r\|B\|h + \delta)\bigr\}. \tag{5.32}$$

From (5.32) and basing on Lemma 4 and 5 we have

$$\Delta[P_\alpha(A)] \geq \frac{1}{38}\omega(r\|B\|h + \delta, r). \tag{5.33}$$

Second case. Suppose that the function $G(\alpha)$ given by relation (5.6) satisfies the inequality

$$G(\alpha) < \frac{1}{18}\|P_\alpha(A)\|(r\|B\|h + \delta). \tag{5.34}$$

Now we construct the finite-dimensional approximation A_h of the operator A. Define the operator $\overline{A_h}$ so that $Sp(\overline{A_h}) = Sp(A)$, and the eigenelements q_n, corresponding to eigenvalues $\sigma_n \in Sp(\overline{A_h})$, defined by the relations

$$q_1 = \frac{e_1 - h_1 e_{i(\alpha)}}{\sqrt{1+h_1^2}},$$

$$q_{i(\alpha)} = \frac{e_{i(\alpha)} + h_1 e_1}{\sqrt{1+h_1^2}},$$

$$q_n = e_n \quad \text{for } n \neq 1 \text{ and } n \neq i(\alpha),$$

where $h_1 = \frac{h}{2\|A\|}$, and e_n are eigenelements of A.

Consider a finite-dimensional subspace H_k, generated by eigenelements e_1, $e_2, \dots, e_k$ of the operator A. We choose the number k so that

$$\sigma_{k+1} < \frac{h}{2} \tag{5.35}$$

and

$$i(\alpha) < k. \tag{5.36}$$

Then we define the operator $A_h \in (H \to H)_2$ as

$$A_h u = \begin{cases} \overline{A_h} u, & \text{for } u \in H_k, \\ 0, & \text{for } u \in H_k^{\perp}. \end{cases} \tag{5.37}$$

Formulas (5.35)–(5.37) yield

$$\|A - A_h\| \leq h. \tag{5.38}$$

Since $e_1 \in H_k$, by setting $\overline{u}_0 = r\|B\|e_1$, $\overline{f}_0 = A\overline{u}_0$ and $\overline{f}_\delta = A\overline{u}_0 + \delta q_{i(\alpha)}$, we obtain $\overline{u}_0 \in M_r \cap H_k$ and

$$\|\overline{f}_\delta - A\bar{u}_0\| \leq \delta. \tag{5.39}$$

From (2.3), (5.38), (5.39) it follows that

$$\Delta[P_\alpha(A)] \geq \|\overline{u}_0 - P_\alpha(A_h)\overline{f}_\delta\|. \tag{5.40}$$

Now we consider the estimate from below for the right-hand side of the inequality (5.40):

$$\|\overline{u}_0 - P_\alpha(A_h)\overline{f}_\delta\| \geq \Big| \|P_\alpha(A_h)\overline{f}_\delta - P_\alpha(A)A\bar{u}_0\| - \|P_\alpha(A)A\overline{u}_0 - \overline{u}_0\| \Big|. \tag{5.41}$$

We estimate each term in the right-hand side of (5.41).

Since $P_\alpha(A_h)q_1 = \lambda_1(\alpha)q_1$ and $P_\alpha(A_h)q_{i(\alpha)} = \lambda_{i(\alpha)}q_{i(\alpha)}$, and taking into account $P_\alpha(A)e_1 = \lambda_1(\alpha)e_1$, where $e_1 = (q_1 + h_1 q_{i(\alpha)})/(1+h_1^2)^{1/2}$, we have

$$\begin{aligned}
&P_\alpha(A_h)\overline{f}_\delta - P_\alpha(A)\overline{f}_0 \\
&\quad = \delta\lambda_{i(\alpha)}q_{i(\alpha)} + \lambda_{i(\alpha)}\frac{r\|A\|\cdot\|B\|h_1}{\sqrt{1+h_1^2}}\cdot q_{i(\alpha)} + \lambda_1(\alpha)\frac{r\|A\|\cdot\|B\|}{\sqrt{1+h_1^2}}\cdot q_1 \\
&\qquad - \lambda_1(\alpha)\frac{r\|A\|\cdot\|B\|}{\sqrt{1+h_1^2}}\cdot q_1 - \lambda_1(\alpha)\frac{r\|A\|\cdot\|B\|h_1}{\sqrt{1+h_1^2}}\cdot q_{i(\alpha)}.
\end{aligned} \tag{5.42}$$

Regrouping the terms in (5.42) and collecting the similar, we obtain

$$\|P_\alpha(A_h)\overline{f}_\delta - P_\alpha(A)A\overline{u}_0\| = \big(\lambda_{i(\alpha)} - \lambda_1(\alpha)\big)\frac{r\|A\|\cdot\|B\|h_1}{\sqrt{1+h_1^2}} + \lambda_{i(\alpha)}\cdot\delta. \tag{5.43}$$

Taking into account $\lambda_{i(\alpha)} \ge \frac{1}{2}\|P_\alpha(A)\|$ and (5.43) we have

$$\|P_\alpha(A_h)\overline{f}_\delta - P_\alpha(A)A\overline{u}_0\| \ge \frac{1}{9}\|P_\alpha(A)\|\big(r\|B\|h + \delta\big). \tag{5.44}$$

Relations (5.34) and (5.44) yield

$$\Delta[P_\alpha(A)] \ge \frac{1}{18}\big\{G(\alpha) + \|P_\alpha(A)\|(r\|B\|h+\delta)\big\}. \tag{5.45}$$

From (5.45) and on the basis of Lemma 4 and 5 we obtain the inequality

$$\Delta[P_\alpha(A)] \ge \frac{1}{18}\omega(r\|B\|h + \delta, r). \tag{5.46}$$

Relations (5.33) and (5.46) prove the theorem. □

6 The establishment method

Let $A \in (H \to H)_1$. Then following [5], we define the operator e^{At} by the formula

$$e^{At} = \sum_{n=0}^{\infty}\frac{A^n\cdot t^n}{n!}. \tag{6.1}$$

Note that the series in (6.1) converges absolutely for all values t and define the linear bounded operator acting from H into H. We shall use further the following result [9].

Lemma 10. *Let A_1, A_2 be nonnegative self-adjoint linear bounded operators acting from H into H such that $\|A_1 - A_2\| \le h$. Then*

$$\|e^{-A_1} - e^{-A_2}\| \le 2h.$$

Assume that $A \in (H \to H)_1$ is nonnegative and self-adjoint. Then, following [6], the establishment method consists in reducing equation (2.1) to the Cauchy problem for differential equation

$$\frac{du(t)}{dt} + Au(t) = f, \quad u(0) = 0. \tag{6.2}$$

From the Hilbert–Schmidt theorem [4] the existence and uniqueness of solution of problem (6.2) for each $t > 0$ and $f \in H$ follows. This solution may be represented as the series

$$\overline{P}_\alpha(A)f = \sum_{n=1}^{\infty} \frac{f_n}{\sigma_n}\left(1 - e^{-\sigma_n t}\right)e_n, \tag{6.3}$$

where e_n are the eigenelements and σ_n the eigenvalues of the operator A and $f_n = (f, e_n)$ for each n.

From formula (6.3) it follows that for each $f_0 = Au_0$, where $u_0 \in H$, the solution $u(t)$ converges to u_0 for $t \to \infty$. Thus, problem (6.2) generates the set of operators $\{P_t : t > 0\}$, regularizing (2.1) in the whole H. Change the variable

$$\alpha = \frac{1}{t}. \tag{6.4}$$

Then the function $\Phi(\sigma, \alpha)$ generating the regularizing set of operators in the establishment method will be as follows:

$$\Phi(\sigma, \alpha) = \frac{\left(1 - e^{-\frac{\sigma}{\alpha}}\right)}{\sigma}. \tag{6.5}$$

Thus, following (6.3)–(6.5), the approximate solution u_δ^α of solution (2.1) for $f = f_\delta$, obtained by the establishment method will be

$$u_\delta^\alpha = \overline{P}_\alpha(A)f_\delta = \sum_{n=1}^{\infty} \frac{\overline{f}_n}{\sigma_n}\left(1 - e^{-\frac{\sigma_n}{\alpha}}\right)e_n, \tag{6.6}$$

where $\overline{f}_n = (f_\delta, e_n)$, and the parameter α is for now not defined.

Let $\overline{A}_h \in (H \to H)_2$ be a nonnegative self-adjoint operator, $\varsigma_1, \varsigma_2, \dots, \varsigma_N$ be eigenvalues and $\mu_1, \mu_2, \dots, \mu_N$ be eigenelements of the operator $\overline{A}_h$. Suppose also that the operator $\overline{A}_h$ satisfies the relation

$$\|\overline{A}_h - A\| \le h. \tag{6.7}$$

Finite-dimensional approximation in the establishment method is the change of problem (6.2) by its finite-dimensional analog

$$\frac{du(t)}{dt} + \overline{A}_h u(t) = \hat{f}_\delta, \quad u(0) = 0, \tag{6.8}$$

where $\hat{f}_\delta = \mathrm{pr}(f_\delta, H_N)$, and $H_N = L\langle \varsigma_1, \varsigma_2, \dots, \varsigma_N \rangle$ is the subspace generated by the system $\varsigma_1, \varsigma_2, \dots, \varsigma_N$.

Problem (6.8) is the Cauchy problem for the system of differential equations which for each $f \in H$ has a unique solution $u^\alpha_{\delta h}(N)$. This solution can be represented in the form

$$\overline{P}_\alpha(\overline{A}_h)\hat{f}_\delta = u^\alpha_{\delta h}(N) = \sum_{n=1}^{N} \frac{\hat{f}_n}{\mu_n}\big(1 - e^{-\frac{\mu_n}{\alpha}}\big)\varsigma_n, \tag{6.9}$$

where $\hat{f}_n = (\hat{f}_\delta, \varsigma_n)$.

The problem we have to solve is to estimate the deviation of the finite-dimensional approximation $u^\alpha_{\delta h}(N)$ from the exact solution u_0 and to choose the regularizing parameter $\alpha = \alpha(\delta, h)$ which minimizes this estimate.

7 The estimate from above for deviation of the finite-dimensional approximation from the exact solution in the class M_r

Let $u_0 \in M_r$, and the set $M_r = B\overline{S_r}$ be defined by the operator $B = g(A)$, where

$$g(A) = A^p, \quad p > 0. \tag{7.1}$$

Lemma 11. *Let $\overline{A}_h \in (H \to H)_2$ be a nonnegative self-adjoint operator satisfying relation* (6.7) *and the operator $\overline{P}_\alpha(\overline{A}_h)$ define the finite-dimensional approximation $u^\alpha_{\delta h}(N)$ given by formula* (6.9). *Then, the following estimate holds:*

$$\|\overline{P}_\alpha(\overline{A}_h)\| \le \frac{1}{\alpha}.$$

Proof. Relation (6.9) yields

$$\|\overline{P}_\alpha(\overline{A}_h)\|^2 = \sup\Big\{\sum_{n=1}^{N} \frac{f_n^2}{\mu_n^2}\big(1 - e^{-\frac{\mu_n}{\alpha}}\big)^2 : \sum_{n=1}^{N} f_n^2 \le 1\Big\}, \tag{7.2}$$

and from the results in [9] we have for each $\alpha > 0$

$$\sup_{\mu > 0}\big(1 - e^{-\frac{\mu}{\alpha}}\big) \le \frac{1}{\alpha}.$$

This, together with (7.2) yields

$$\left\|\overline{P}_\alpha(\overline{A}_h)\right\| \leq \frac{1}{\alpha}$$

which proves the lemma. □

To estimate the error of the finite-dimensional approximation we introduce the function

$$\Delta_1(\alpha) = \sup\left\{\|u_0 - P_\alpha(A)Au_0 : u_0 \in M_r\right\}. \tag{7.3}$$

Lemma 12. *Under the conditions formulated above for the class of uniform regularization and for operators A and $\overline{P}_\alpha(A)$ the following estimate holds:*

$$\Delta_1(\alpha) \leq r\left(\frac{p}{e}\right)^p \alpha^p.$$

Proof. Relations (6.3) and (7.3) yield

$$\Delta_1^2(\alpha) = \sup\left\{\sum_{n=1}^{\infty} \sigma_n^{2p} e^{-\frac{2\sigma_n}{\alpha}} v_n^2 : \sum_{n=1}^{\infty} v_n^2 \leq 1\right\}, \tag{7.4}$$

and from the results in [9] we have

$$\sup_{x>0}\left(\frac{x^{2p}}{e^x}\right) \leq \left(\frac{2p}{e}\right)^{2p}.$$

This inequality and (7.4) prove the lemma. □

Let C_1 be a certain positive number and

$$\overline{\alpha} = \overline{\alpha}(\delta, h) = C_1\left(r\|B\|h + \delta\right)^{\frac{1}{p+1}}. \tag{7.5}$$

Theorem 2. *Let the finite-dimensional approximation $u_{\delta h}^{\overline{\alpha}}(N)$ be defined by formula* (6.9)*, the regularization parameter $\overline{\alpha} = \overline{\alpha}(\delta, h)$ be defined by* (7.5)*, and the exact solution $u_0 \in M_r$, where M_r be defined by* (7.1)*. Then the following estimate holds:*

$$\left\|u_{\delta h}^{\overline{\alpha}}(N) - u_0\right\| \leq C_2\left(r\|B\|h + \delta\right)^{\frac{p}{p+1}},$$

where C_2 is a certain positive constant.

Proof. Since $\overline{P}_{\overline{\alpha}}(\overline{A}_h)\hat{f}_\delta = \overline{P}_{\overline{\alpha}}(\overline{A}_h)f_\delta$, we have

$$\begin{aligned}\left\|u^{\overline{\alpha}}_{\delta h}(N) - u_0\right\| \le \|\overline{P}_{\overline{\alpha}}(\overline{A}_h)f_\delta - \overline{P}_{\overline{\alpha}}(\overline{A}_h)\overline{A}_h u_0\| \\ + \|\overline{P}_{\overline{\alpha}}(\overline{A}_h)\overline{A}_h u_0 - \overline{P}_{\overline{\alpha}}(A)Au_0\| \\ + \|\overline{P}_{\overline{\alpha}}(A)Au_0 - u_0\|.\end{aligned} \tag{7.6}$$

We estimate each term in the left-hand side of (7.6). From Lemma 12 and (7.5) we conclude

$$\|\overline{P}_{\overline{\alpha}}(A)Au_0 - u_0\| \le r\left(\frac{p}{e}\right)^p C_1 \cdot (r\|B\|h + \delta)^{\frac{p}{p+1}}. \tag{7.7}$$

From

$$\|\overline{P}_{\overline{\alpha}}(\overline{A}_h)f_\delta - \overline{P}_{\overline{\alpha}}(\overline{A}_h)\overline{A}_h u_0\| \le \|\overline{P}_{\overline{\alpha}}(\overline{A}_h)\| \cdot (r\|B\|h + \delta)$$

and from Lemma 11 it follows that

$$\|\overline{P}_{\overline{\alpha}}(\overline{A}_h)f_\delta - \overline{P}_{\overline{\alpha}}(\overline{A}_h)\overline{A}_h u_0\| \le \frac{1}{C_1} \cdot (r\|B\|h + \delta)^{\frac{p}{p+1}}. \tag{7.8}$$

Since

$$\|\overline{P}_{\overline{\alpha}}(\overline{A}_h)\overline{A}_h u_0 - \overline{P}_{\overline{\alpha}}(A)Au_0\| \le \|e^{A/\overline{\alpha}} - e^{\overline{A}_h/\overline{\alpha}}\| \cdot \|u\| \tag{7.9}$$

and taking into account Lemma 10, relations (7.5), (7.9) and $u_0 \in M_r$, we obtain

$$\|\overline{P}_{\overline{\alpha}}(\overline{A}_h)\overline{A}_h u_0 - \overline{P}_{\overline{\alpha}}(A)Au_0\| \le \frac{2}{C_1} \cdot (r\|B\|h + \delta)^{\frac{p}{p+1}}. \tag{7.10}$$

From (7.6)–(7.8) and (7.10) it follows that a positive number $C_2 > 0$ exists such that

$$\left\|u^{\overline{\alpha}}_{\delta h}(N) - u_0\right\| \le C_2\left(r\|B\|h + \delta\right)^{\frac{p}{p+1}},$$

which proves the theorem. □

From a theorem in [3] it follows that for $g(A) = A^p$ the continuity modules of the inverse operator is defined by the formula

$$\omega(\tau, r) = r^{\frac{1}{p+1}}\tau^{\frac{p}{p+1}}. \tag{7.11}$$

Thus combining Theorems 1, 2 and formula (7.11) we can formulate the following theorem.

Theorem 3. *Let the regularization parameter* $\overline{\alpha} = \overline{\alpha}(\delta, h)$ *be defined by* (7.5), *and* $u_0 \in M_r$ *be the exact solution, where* M_r *is defined by* (7.1). *Then for the establishment method, the following estimate, exact by the order, holds:*

$$\Delta[\overline{P}_{\overline{\alpha}}(A)] = C_2\left(r\|B\|h + \delta\right)^{\frac{p}{p+1}}.$$

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