A NON-STATIONARY MODEL OF THE INCOMPRESSIBLE VISCOELASTIC KELVIN–VOIGT FLUID OF NON-ZERO ORDER IN THE MAGNETIC FIELD OF THE EARTH

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We investigate the Cauchy–Dirichlet problem for a system of Oskolkov equations of nonzero order. The considered mathematical model describes the flow of an incompressible viscoelastic Kelvin–Voigt fluid in the magnetic field of the Earth. The model takes into account that the fluid is subject to various external influences, which depend on both the coordinate of the point in space and the time. The first part of the paper presents the known results obtained by the authors earlier and based on the theory of solvability of the Cauchy problem for semilinear nonautonomous Sobolev type equations. In the second part, we reduce the considered mathematical model to an abstract Cauchy problem. In the third part, we prove the main result that is the theorem on the existence and uniqueness of the solution. Also, we establish the conditions for the existence of quasi-stationary semitrajectories, and describe the extended phase space of the model under study. In this paper, we summarize our results for the Oskolkov system that simulates the motion of a viscoelastic incompressible Kelvin–Voigt fluid of zero order in the magnetic field of the Earth.

Keywords: magnetohydrodynamics; Sobolev type equations; extended phase space; incompressible viscoelastic fluid.

Introduction

The Oskolkov's system of equations

$$(1 - \varkappa \nabla^2)v_t = \nu \nabla^2 v - (v \cdot \nabla)v + \sum_{l=1}^{K} \beta_l \nabla^2 w_l - \frac{1}{\rho}\nabla p - 2\Omega \times v + \frac{1}{\rho\mu}(\nabla \times b) \times b + f^1,$$

$$\nabla \cdot v = 0, \quad \nabla \cdot b = 0, \quad b_t = \delta\nabla^2 b + \nabla \times (v \times b) + f^2, \tag{1}$$

$$\frac{\partial w_l}{\partial t} = v + \alpha_l w_l, \quad \alpha_l \in \mathbb{R}_-, \quad \beta_l \in \mathbb{R}_+, \quad l = \overline{1, K},$$

simulates the flow of an incompressible viscoelastic Kelvin–Voigt fluid [1] of non-zero order K in the magnetic field of the Earth. Here the vector functions $v = (v_1(x,t), v_2(x,t), \ldots, v_n(x,t))$ and $b = (b_1(x,t), b_2(x,t), \ldots, b_n(x,t))$ characterize fluid velocity and magnetic induction, respectively, $p = p(x,t)$ is the pressure, $\varkappa$ is the coefficient of elasticity, ν is the coefficient of viscosity, Ω is the corner velocity, δ is the magnetic viscosity, μ is the magnetic permeability, ρ is the density, and the parameters β_l, $l = \overline{1, K}$ determine the time of pressure retardation (delay). The absolute terms $f^1 = (f^1_1, \ldots, f^1_n)$, $f^1_i = f^1_i(x,t), f^2 = f^2(x,t)$ correspond to external influences on the fluid.

Consider the first initial boundary value problem for system (1):

$$v(x,0) = v_0(x), \quad b(x,0) = b_0(x), \quad w_l(x,0) = w_{l0}(x) \quad x \in D,$$

$$v(x,t) = 0, \quad b(x,t) = 0, \quad w_l(x,t) = 0 \quad (x,t) \in \partial D \times \mathbb{R}_+, \quad l = \overline{1, K} \tag{2}$$

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under assumption that $\mu = 1$ and $\rho = 1$. Here $D \subset \mathbb{R}^n$ is a bounded domain with the boundary ∂D of the class C^∞.

Problems that are similar to problem (1), (2) take place, for example, in mathematical modelling in geophysical sciences [2].

Note that degenerate models of magnetohydrodynamics were previously studied by the authors in the papers [3–6]. A distinctive feature of the present paper is the presence of the vector-functions $f^1 = (f_1^1, \ldots, f_n^1)$, $f_i^1 = f_i^1(x, t)$, $f^2 = f^2(x, t)$ in the right hand side of equation (1). The paper [7] considers the model of magnetohydrodynamics, which takes into account various external influences for $K = 0$. The case of $K > 0$ is investigated for the first time.

Problem (1), (2) is investigated in the framework of the theory of semilinear Sobolev type equations [8, 9]. The main tool of the study is the notion of a relative p-sectorial operator and a resolving degenerate semigroup of operators generated by this operator [10, 11]. We prove the theorem on the existence and uniqueness of the solution to this problem, and describe the extended phase space of the problem.

The article consists of three sections. Section 1 gives the known necessary results of the theory of semi-linear Sobolev type equations [10, 12]. Section 2 reduces problem (1), (2) to the Cauchy problem for the semi-linear Sobolev type equation. Section 3 presents the theorem on the existence and uniqueness of the solution to the considered problem, shows that the solution is a quasi-stationary trajectory, and describes the extended phase space of the problem.

1. Semi-Linear Non-Stationary Sobolev Type Equations

Let $\mathcal{U}$ and $\mathcal{F}$ be Banach spaces, the operator $L \in \mathcal{L}(\mathcal{U}; \mathcal{F})$, i.e. L is linear and continuous, the operator $M : \text{dom}\, M \to \mathcal{F}$ be linear, closed and densely defined in $\mathcal{U}$, i.e. $M \in \mathcal{Cl}(\mathcal{U}; \mathcal{F})$. Denote $\mathcal{U}_M = \{u \in \text{dom}\, M : \|u\| = \|Mu\|_{\mathcal{F}} + \|u\|_{\mathcal{U}}\}$. Let the operator $F \in C^\infty(\mathcal{U}_M; \mathcal{F})$. We suppose that the operator $F \in \mathcal{C}^\infty(\mathcal{U}_M; \mathcal{F})$, and the function $f \in \mathcal{C}^\infty(\mathbb{R}_+; \mathcal{F})$.

Consider the Cauchy problem
$$u(0) = u_0 \tag{3}$$
for the semi-linear non-stationary Sobolev type equation
$$L\dot{u} = Mu + F(u) + f(t). \tag{4}$$

By a *local solution* (hereinafter, *solution*) to problem (3), (4) we mean the vector function $u \in \mathcal{C}^\infty((0, T); \mathcal{U}_M)$ that satisfies equation (4) and such that $u(t) \to u_0$ for $t \to 0+$.

Let the operator M be strongly (L, p)-sectorial [12]. It is well known that, under this condition, problem (3), (4) can have several solutions [13]. Therefore, we are interested in only such solutions to problem (3), (4) that are *quasi-stationary semitrajectories*.

Definition 1. *Suppose that the space $\mathcal{U}$ splits into the direct sum $\mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1$ such that $\ker L \subset \mathcal{U}_0$. A solution $u = v + w$ to equation (4), where $v(t) \in \mathcal{U}_0$ and $w(t) \in \mathcal{U}_1$ for all $t \in (0, T)$, is called a quasi-stationary semitrajectory, if $L\dot{v} \equiv 0$.*

Also, it is known [12] that problem (3), (4) can has no solutions for some $u_0 \in \mathcal{U}_M$. Therefore, we introduce another definition.

Definition 2. *The set $\mathcal{B}^t \subset \mathcal{U}_M \times \bar{\mathbb{R}}_+$ is called the extended phase space of equation (4), if for any point $u_0 \in \mathcal{U}_M$ such that $(u_0, 0) \in \mathcal{B}^0$ there exists a unique solution to problem (3), (4), and $(u(t), t) \in \mathcal{B}^t$.*

We consider problem (3), (4) under the condition that the operator M is strongly (L, p)-sectorial [12]. In this case, the problem can have no solution for some $u_0 \in \mathcal{U}_M$, and even if there exists a solution for all $u_0 \in \mathcal{U}_M$, then the solution can be non-unique.

It is well known that if the operator M is strongly (L, p)-sectorial, then $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1$, $\mathcal{F} = \mathcal{F}^0 \oplus \mathcal{F}^1$, where $\mathcal{U}^0 = \{\varphi \in \mathcal{U} : U^t \varphi = 0 \ \exists t \in \mathbb{R}_+\}, \mathcal{F}^0 = \{\psi \in \mathcal{F} : F^t \psi = 0 \ \exists t \in \mathbb{R}_+\}$ are the *kernels*, and $\mathcal{U}^1 = \{u \in \mathcal{U} : \lim_{t \to 0+} U^t u = u\}, \mathcal{F}^1 = \{f \in \mathcal{F} : \lim_{t \to 0+} F^t f = f\}$ are the *images* of the analytic solving semigroups

$$U^t = \frac{1}{2\pi i} \int_\Gamma R_\mu^L(M) e^{\mu t} d\mu, F^t = \frac{1}{2\pi i} \int_\Gamma L_\mu^L(M) e^{\mu t} d\mu \tag{5}$$

of the linear homogeneous equation

$$L\dot{u} = Mu, \tag{6}$$

where $\Gamma \subset S_{\Theta, a}^L(M)$ is a contour such that $\arg \mu \to \pm\Theta$ for $|\mu| \to +\infty$.

Denote by L_k and M_k the restrictions of the operators L and M on $\mathcal{U}^k (\mathcal{U}^k \cap \operatorname{dom} M)$, $k = 0, 1$, respectively. Then $L_k : \mathcal{U}^k \to \mathcal{F}^k$, $M_k : \mathcal{U}^k \cap \operatorname{dom} M \to \mathcal{F}^k$, $k = 0, 1$, and the restrictions M_0 and L_1 of the operators M and L on the spaces $\mathcal{U}^0 \cap \operatorname{dom} M$ and $\mathcal{U}^1$ are linear continuous operators and have bounded inverse operators.

Therefore, problem (3), (4) is reduced to an equivalent system, which we call the *normal form* of problem (3), (4):

$$\begin{aligned} R\dot{u}^0 &= u^0 + G(u) + g(t), \quad u^0(0) = u_0^0, \\ \dot{u}^1 &= Su^1 + H(u) + h(t) \quad u^1(0) = u_0^1, \end{aligned} \tag{7}$$

where $u^k \in \mathcal{U}^k$, $k = 0, 1$, $u = u^0 + u^1$, the operators $R = M_0^{-1} L_0$, $S = L_1^{-1} M_1$, $G = M_0^{-1}(I - Q)F$, $H = L_1^{-1} QF$, $g = M_0^{-1}(I - Q)f$, $h = L_1^{-1} Qf$.

Here $Q \in \mathcal{L}(F)(\equiv \mathcal{L}(F; F))$ is the projector that splits the space $\mathcal{F}$ as required.

Further, we study only the quasi-stationary semitrajectories of equation (4), for which $R\dot{u}^0 \equiv 0$. To this end, we assume that the operator R is *bi-splitting*, i.e. the kernel $\ker R$ and the image $\operatorname{im} R$ are completed in the space $\mathcal{U}$. Suppose that $\mathcal{U}^{00} = \ker R$. Denote by $\mathcal{U}^{01} = \mathcal{U}^0 \ominus \mathcal{U}^{00}$ a complement of the subspace $\mathcal{U}^{00}$. Then the first equation of normal form (7) is reduced to

$$R\dot{u}^{01} = u^{00} + u^{01} + G(u) + g(t), \tag{8}$$

where $u = u^{00} + u^{01} + u^1$.

Theorem 1. *Let the operator M be strongly (L, p)-sectorial, and the operator R be bi-splitting. Suppose that there exists the quasi-stationary semitrajectory $u = u(t)$ of equation (4). Then $u = u(t)$ satisfies the following relations:*

$$0 = u^{00} + u^{01} + G(u) + g(t), u^{01} = \text{const.} \tag{9}$$

It is known that if the operator M is strongly (L, p)-sectorial, then the operator S is sectorial. Therefore, on $\mathcal{U}^1$, the operator S generates an analytic semigroup, which we denote by $\{U_1^t : t \geq 0\}$, since the operator U_1^t is a restriction of the operator U^t on $\mathcal{U}^1$.

Since $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1$, then there exists the projector $P \in \mathcal{L}(\mathcal{U})$ corresponding to this splitting. It is easy to see that $P \in \mathcal{L}(\mathcal{U}_M)$. Then the space $\mathcal{U}_M$ splits into the direct sum $\mathcal{U}_M = \mathcal{U}_M^0 \oplus \mathcal{U}_M^1$ such that the embedding $\mathcal{U}_M^k \subset \mathcal{U}^k$, $k = 0, 1$, is dense and continuous. Further, denote by A_v' the Frechet derivative at the point $v \in \mathcal{V}$ of the operator A defined on the Banach space $\mathcal{V}$.

Theorem 2. *Let the operator M be strongly (L, p)-sectorial, the operator R be bi-splitting, the operator $F \in \mathcal{C}^\infty(\mathcal{U}_M; \mathcal{F})$, and the vector-function $f \in \mathcal{C}^\infty(\mathbb{R}_+; \mathcal{F})$. Suppose that the following conditions are fulfilled.*

(i) *In the neighborhood $\mathcal{O}_{u_0} \subset \mathcal{U}_M$ of the point u_0, the following relation takes place:*

$$0 = u_0^{01} + (I - P_R)(G(u^{00} + u_0^{01} + u^1) + g(t)). \tag{10}$$

(ii) *The projector $P_R \in \mathcal{L}(\mathcal{U}_M^0)$, and the operator $I + P_R G'_{u_0^0} : \mathcal{U}_M^{00} \to \mathcal{U}_M^{00}$ is the topological linear isomorphism $(\mathcal{U}_M^{00} = \mathcal{U}_M \cap \mathcal{U}^{00})$.*

(iii) *For the analytic semigroup $\{U_1^t : t \geq 0\}$, the following condition is fulfilled:*

$$\int\limits_0^\tau \|U_1^t\|_{\mathcal{L}(\mathcal{U}^1; \mathcal{U}_M^1)} dt < \infty \quad \forall \tau \in \mathbb{R}_+. \tag{11}$$

Then there exists the unique solution to problem (3), (4), *which is the quasi-stationary semitrajectory.*

Remark 1. Condition (11) is not satisfied for ordinary analytic semigroups having the estimate $\|U_1^t\|_{\mathcal{L}(\mathcal{U}^1; \mathcal{U}_M^1)} < const/t$. Denote by $\mathcal{U}_\alpha^1 = [\mathcal{U}^1; \mathcal{U}_M^1]_\alpha$, $\alpha \in [0, 1]$, some interpolation space constructed by the operator S. Complete the condition $F \in \mathcal{C}^\infty(\mathcal{U}_M^1; \mathcal{F})$ of Theorem 2 with the condition $H \in \mathcal{C}^\infty(\mathcal{U}_M^1; \mathcal{U}_\alpha^1)$, and replace Condition (11) with

$$\int\limits_0^\tau \|U_1^t\|_{\mathcal{L}(\mathcal{U}^1; \mathcal{U}_\alpha^1)} dt < \infty, \quad \tau \in \mathbb{R}_+. \tag{12}$$

Then the statement of Theorem 2 is the same.

Let $\mathcal{U}_k$ and $\mathcal{F}_k$ be Banach spaces, the operators $A_k \in \mathcal{L}(\mathcal{U}_k, \mathcal{F}_k)$, and the operators $B_k : \operatorname{dom} B_k \to \mathcal{F}$ be linear and closed with domain of definitions $\operatorname{dom} B_k$, which are dense in $\mathcal{U}_k$, $k = 1, 2$. Construct the spaces $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and the operators $L = A_1 \otimes A_2$, $M = B_1 \otimes B_2$. By the construction, the operator $L \in \mathcal{L}(\mathcal{U}; \mathcal{F})$, and the operator $M : \operatorname{dom} M \to \mathcal{F}$ is linear, closed and densely defined, $\operatorname{dom} M = \operatorname{dom} B_1 \times \operatorname{dom} B_2$.

Theorem 3. *Let the operators B_k be strongly (A_k, p_k)-sectorial, $k = 1, 2$, and $p_1 \geq p_2$. Then the operator M is strongly (L, p_1)-sectorial.*

2. Reduction to Abstract Cauchy Problem

In order to reduce problem (1), (2) to problem (3), (4), we transfer from system (1) to the system

$$(1 - \varkappa\nabla^2)v_t = \nu\nabla^2 v - (v \cdot \nabla)v + \sum_{l=1}^K \beta_l \nabla^2 w_l - \bar{p} - 2\Omega \times v + (\nabla \times b) \times b + f^1,$$

$$\nabla(\nabla \cdot v) = 0, \quad \nabla(\nabla \cdot b) = 0, \quad b_t = \delta\nabla^2 b + \nabla \times (v \times b) + f^2. \tag{13}$$

$$\frac{\partial w_l}{\partial t} = v + \alpha_l w_l, \quad \alpha_l \in \mathbb{R}_-, \quad \beta_l \in \mathbb{R}_+, \quad l = \overline{1, K}.$$

We are interested in solvability of problem (13), (2). Following the paper [12], we introduce the spaces $\mathbf{H}_\sigma^2$, $\mathbf{H}_\pi^2$, $\mathbf{H}_\sigma$, and $\mathbf{H}_\pi$. Here $\mathbf{H}_\sigma^2$ and $\mathbf{H}_\sigma$ are subspaces of the solenoid functions in the spaces $(W_2^2(D))^n \cap \cap (\overset{\circ}{W}{}_2^1(D))^n$ and $(L_2(D))^n$, respectively, and $\mathbf{H}_\pi^2$ and $\mathbf{H}_\pi$ are their orthogonal (in the sense of $(L_2(D))^n$) complements. Denote by Σ both the orthoprojector on $\mathbf{H}_\sigma$ and its

restriction on the space $(W_2^2(D))^n \cap (\overset{\circ}{W}{}_2^1(D))^n$. Suppose that $\Pi = I - \Sigma$. The equality $A = \nabla^2 E_n : \mathbf{H}_\sigma^2 \oplus \mathbf{H}_\pi^2 \to \mathbf{H}_\sigma \oplus \mathbf{H}_\pi$, where E_n is a unit matrix of order n, defines a linear continuous matrix operator with discrete finite-multiple spectrum $\sigma(A) \subset \mathbb{R}$ that tends only to $-\infty$. The formula $B_v : v \to \nabla(\nabla \cdot v)(B_b : b \to \nabla(\nabla \cdot b))$ gives the linear continuous surjective operator $B_v(B_b) : \mathbf{H}_\sigma^2 \oplus \mathbf{H}_\pi^2 \to \mathbf{H}_\pi$ with the kernel $\ker B_v = B_b = \mathbf{H}_\sigma^2$. We use the natural isomorphism of the direct sum and the Cartesian product of Banach spaces in order to introduce the spaces $\mathcal{U}_{10} = \mathbf{H}_\sigma^2 \times \mathbf{H}_\pi^2 \times \mathbf{H}_p$, $\mathcal{F}_{10} = \mathbf{H}_\sigma \times \mathbf{H}_\pi \times \mathbf{H}_p$, where $\mathbf{H}_p = \mathbf{H}_\pi$; $\mathcal{U}_{1i} = \mathbf{H}^2 \cap \overset{\circ}{\mathbf{H}}{}^1 = \mathbf{H}_\sigma^2 \times \mathbf{H}_\pi^2$, and $\mathcal{F}_{1i} = \mathbf{L}_2 = \mathbf{H}_\sigma \times \mathbf{H}_\pi$, $i = \overline{1, K}$. Then spaces $\mathcal{U}_1 = \oplus_{l=0}^K \mathcal{U}_{1l}$, $\mathcal{F}_1 = \oplus_{l=0}^K \mathcal{F}_{1l}$.

The operators A_1 and B_1 are defined by the formulas $A_1 = \operatorname{diag}[\hat{A}_1, E_k]$, where

$$\hat{A}_1 = \begin{pmatrix} \check{A}_1 & O \\ O & O \end{pmatrix}, \quad \check{A}_1 = \begin{pmatrix} \Sigma(I - \lambda A)\Sigma & \Sigma A(I - \lambda A)\Pi \\ \Pi(I - \lambda A)\Sigma & \Pi A(I - \lambda A)\Pi \end{pmatrix};$$

$B_1 = (B_1^{ij})_{i,j=1}^2$, where

$$B_1^{11} = \begin{pmatrix} \nu\Sigma A & \nu\Sigma A & O \\ \nu\Pi A & \nu\Pi A & -I \\ O & B & O \end{pmatrix}, \quad B_1^{12} = \begin{pmatrix} \beta_1\Sigma A & \dots & \beta_K\Sigma A \\ \beta_1\Pi A & \dots & \beta_K\Pi A \\ O & \dots & O \end{pmatrix}.$$

In the matrix B_1^{11}, $B = \nabla(\nabla \cdot v) - \nabla(\nabla \cdot b) = B_v - B_b$. The matrix B_1^{21} contains K rows of the form (I, I, O), $B_1^{22} = \operatorname{diag}[\alpha_1, \dots, \alpha_K]$.

Remark 2. Denote by A_σ the restriction of the operator ΣA on $\mathbf{H}_\sigma^2$. According to the Solonnikov–Vorovich–Yudovich theorem, the spectrum $\sigma(A_\sigma)$ is real, discrete, finite-multiple, and tends only to $-\infty$.

Theorem 4.

i) The operators A_1, B_1 belong to $\mathcal{L}(\mathcal{U}_1; \mathcal{F}_1)$, and if $\varkappa^{-1} \notin \sigma(A)$, then the operator A_1 is bi-splitting, $\ker A_1 = \{0\} \times \{0\} \times \mathbf{H}_p \times \underbrace{\{0\} \times \dots \times \{0\}}_K$, $\operatorname{im} A_1 = \mathbf{H}_\sigma \times \mathbf{H}_\pi \times \{0\} \times \mathcal{F}_1 \times \mathcal{F}_2 \times \dots \times \mathcal{F}_K$.

ii) If $\lambda^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$, then the operator B_1 is $(A_1, 1)$-bounded.

Proof. The statement of the theorem is the direct corrolary of the results obtained in [12]. $\square$

Remark 3. The (L, p)-bounded operator is defined, for example, in [12].

Suppose that $\mathcal{U}_2 = \mathcal{F}_2 = L_2(D)$. The equality $B_2 = \delta\nabla^2 : \operatorname{def} B_2 \to \mathcal{F}_2$ defines the linear closed and densely defined operator B_2, $\operatorname{dom} B_2 = W_2^2(D) \cap \overset{\circ}{W}{}_2^1(D)$. Let $A_2 \equiv I$.

Theorem 5. *The operator B_2 is strongly A_2-sectorial.*

Proof. The statement of the theorem follows from the sectoriality of the operator B_2 [14]. $\square$

Let $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$.

The vector u of the space $\mathcal{U}$ has the form $u = \operatorname{col}(u_\sigma, u_\pi, u_p, w_1, \dots, w_K, u_b)$, where $\operatorname{col}(u_\sigma, u_\pi, u_p, w_1, \dots, w_K) \in \mathcal{U}_1$, and $u_b \in \mathcal{U}_2$, $u_b = (b_\sigma, b_\pi)$, $b_\sigma \in \mathbf{H}_\sigma^2$, $b_\pi \in \mathbf{H}_\pi^2$. The vector $f \in \mathcal{F}$ has the similar form. Define the operators L and M by the equalities $L = A_1 \otimes A_2$ and $M = B_1 \otimes B_2$. The operator L belongs to $\mathcal{L}(\mathcal{U}; \mathcal{F})$, and the operator $M : \operatorname{dom} M \to \mathcal{F}$ is linear, closed and densely defined, $\operatorname{dom} M = \mathcal{U}_1 \times \operatorname{dom} B_2$.

Theorem 6. *Let $\varkappa^{-1} \notin \sigma(A)$, then the operator M is strongly $(L, 1)$-sectorial.*

Proof. By virtue of Theorem 4 and the results of Paragraph 3.1. [12], the operator B_1 is strongly $(A_1, 1)$-sectorial. Therefore, taking into account Theorems 3 and 5, we obtain that the statement of the theorem is true. □

Let us construct the nonlinear operator F. Represent the operator as

$$F = F_1 \otimes F_2,$$

where

$$F_1 = F_1(u_\sigma, u_\pi, b) = col(-\Sigma(((u_\sigma + u_\pi) \cdot \nabla)(u_\sigma + u_\pi) - 2\Omega \times (u_\sigma + u_\pi) + (\nabla \times b) \times b + f^1),$$

$$-\Pi(((u_\sigma + u_\pi) \cdot \nabla)(u_\sigma + u_\pi) - 2\Omega \times (u_\sigma + u_\pi) + (\nabla \times b) \times b + f^1), \underbrace{0, \ldots, 0}_{K+1}),$$

$$F_2 = F_2(u_\sigma, u_\pi, b) = \nabla \times ((u_\sigma + u_\pi) \times b) + f^2.$$

In our case, $\mathcal{U}_M = \mathcal{U}_1 \times \operatorname{dom} B_2$, since the operator B_1 is continuous.

Theorem 7. *The operator F belongs to $\mathcal{C}^\infty(\mathcal{U}_M; \mathcal{F})$.*

Proof. The statement of the theorem follows from the fact that for any $u \in \mathcal{U}_M$ the operator F'_u belongs to $\mathcal{L}(\mathcal{U}_M; \mathcal{F})$, the second Frechet derivative F'''_u of the operator F is the continuous bilinear operator that belongs to $\mathcal{U}_M \times \mathcal{U}_M$ in $\mathcal{F}$, and $F'''_u \equiv O$ (similarly to [12]). □

Therefore, we have reduced (1), (2) to (3), (4), and we can consider these two problems to be equivalent. Let us verify the conditions of Theorems 1 and 2.

3. Theorem on Existence and Uniqueness of Solution

By virtue of Theorem 6 and the results of Paragraph 3.1. [12], there exists the analytic semigroup $\{U^t : t \in \mathbb{R}_+\}$ of the resolving operators of equation (6). In this case, U^t is naturally represented as $U^t = V^t \times W^t$, where $V^t(W^t)$ is the restriction of the operator U^t on $\mathcal{U}_1(\mathcal{U}_2)$. Since B_2 is sectorial, then $W^t = \exp(tB_2)$, and, therefore, the kernel of this semigroup is $\mathcal{W}^\circ = \{0\}$, and the image of this semigroup is $\mathcal{W}^1 = \mathcal{U}_2$.

Consider the semigroup $\{V^t : t \in \mathbb{R}_+\}$. By virtue of Theorems 4 and 6 and results of Paragraph 3.1. [12], this semigroup is extended to the group $\{V^t : t \in \mathbb{R}\}$. The kernel of the semigroup is $\mathcal{V}^0 = \mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01}$, where $\mathcal{U}_1^{00} = \{0\} \times \{0\} \times \mathbf{H}_p \times \{0\} \times \ldots \times \{0\}(= \ker A_1$ due to Theorem 5), and $\mathcal{U}_1^{01} = \Sigma A_\varkappa^{-1} A_{\varkappa\pi}^{-1}[\mathbf{H}_\pi^2] \times \mathbf{H}_\pi^2 \times \underbrace{\{0\} \times \ldots \times \{0\}}_{K+1}$. Here $A_\varkappa = I - \varkappa A$,

$A_{\varkappa\pi}$ is the restriction of the operator $\Pi A_\varkappa^{-1}$ on $\mathbf{H}_\pi$. It is known that if $\varkappa^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$, then the operator $A_{\varkappa\pi} : \mathbf{H}_\pi \to \mathbf{H}_\pi^2$ is topological linear isomorphism [12]. Denote by $\mathcal{U}_1^1$ the image of $\mathcal{V}^1$. Then, since the operator B_1 is strongly $(A_1, 1)$-sectorial, then the space $\mathcal{U}_1$ decomposes into the direct sum of the subspaces $\mathcal{U}_1 = \mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01} \oplus \mathcal{U}_1^1$.

Construct the operator R (see (5), (6)). In our case, $R = B_{10}^{-1} A_{10} \in \mathcal{L}(\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01})$, where $A_{10}(B_{10})$ is the restriction of the operator $A_1(B_1)$ on $\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01}$. Note that the operator B_{10}^{-1} exists due to Theorem 6 and the corresponding results obtained in [12]. By construction, $\ker R = \mathcal{U}_1^{00}$, and the paper [15] shows that $\operatorname{im} R = \mathcal{U}_1^{01}$. Therefore, the operator R is bi-splitting. Denote by P_R the projector of the space $\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01}$ on $\mathcal{U}_1^{00}$ along $\mathcal{U}_1^{01}$. Taking into account the structure of the space $\mathcal{U}_M$, we obtain that the projector P_R belongs to $\mathcal{L}(\mathcal{U}_M^0)$, where $\mathcal{U}_M^0 = \mathcal{U}_M \cap (\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01})(\equiv \mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01})$. Therefore, the following lemma is valid.

Lemma 1. *Suppose that $\varkappa^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$. Then the operator R is bi-splitting, and* $P_R \in \mathcal{L}(\mathcal{U}_M^0)$.

Consider the projectors

$$P_k = \operatorname{diag}[\hat{P}_k, 0], \quad Q_k = \operatorname{diag}[\hat{Q}_k, 0], \quad k = 0, 1,$$

see [12] for a detailed description of these projectors. Taking into account the results of [12] and the fact that the kernel $\mathcal{W}^0 = \{0\}$, we obtain that $I - P = (P_0 + P_1) \times O, \quad Q = (I - Q_0 - Q_1) \times I, \quad P : \mathcal{U} \to \mathcal{U}^1, \quad Q : \mathcal{F} \to \mathcal{F}^1$.) Then, apply the projector IP to equation (4) in our situation and obtain the equations

$$\Pi(\nu A(u_\sigma + u_\pi) - ((u_\sigma + u_\pi) \cdot \nabla)(u_\sigma + u_\pi) + \sum_{l=1}^{K} \beta_l \nabla^2 w_l - u_p - 2\Omega \times (u_\sigma + u_\pi) + \tag{14}$$

$$(\nabla \times b) \times b + f^1(t)) = 0, \quad B\, u_\pi = 0, \quad Bb_\pi = 0.$$

Hence, by virtue of Theorem 1 and the properties of the operator B, we obtain the necessary condition for existence of the quasi-stationary trajectory $u_\pi \equiv 0$, $b_\pi \equiv 0$, i.e. all solutions to problem (2), (13) (if they exist) necessarily belong to the plane $\mathcal{B} = \{u \in \mathcal{U}_M : u_\pi = 0, b_\pi = 0\}$.

Since $\Pi u_p = u_p$, we obtain relation (9) from the first equation of (14), i.e. in our case,

$$u_p = \Pi(\nu A u_\sigma - (u_\sigma \cdot \nabla)u_\sigma + \sum_{l=1}^{K} \beta_l \nabla^2 w_l - 2\Omega \times u_\sigma + (\nabla \times b_\sigma) \times b_\sigma + f^1(t)). \tag{15}$$

Lemma 2. *Under the conditions of Lemma 1, any solution to problem (1), (2) belongs to the set*

$$\mathfrak{M} = \{u \in \mathcal{U}_M : u_\pi = 0, b_\pi = 0, u_p = \Pi(\nu A_\sigma - (u_\sigma \cdot \nabla)u_\sigma + \sum_{l=1}^{K} \beta_l \nabla^2 w_l - 2\Omega \times u_\sigma +$$

$$+(\nabla \times b_\sigma) \times b_\sigma) + f^1(t)\}.$$

Remark 4. Relation (15) gives condition A_2) of Theorem 2 for any point $u_0^0 \in \mathcal{U}_M^{00} (\equiv \mathcal{U}_1^{00} \times \{0\})$. Therefore, similarly to [12], we obtain that the set $\mathfrak{M}$ is a simple Banach manifold that is $\mathcal{C}^\infty$-diffeomorphic to the subspace $\mathcal{U}_1^1 \times \mathcal{U}_2$, and can be the extended phase space of problem (1), (2) ((13), (2)).

Let us verify conditions (11), (12). Construct the space $\mathcal{U}_\alpha = \mathcal{U}_1 \times \overset{\circ}{W}{}_2^1(D)$. Obviously, this space is the interpolation space for the pair $[\mathcal{U}, \mathcal{U}_M]_\alpha$, and $\alpha = 1/2$. As noted above, the semigroup $\{U^t : t \in \mathbb{R}_+\}$ is extended to the group $\{V_1^t : t \in \mathbb{R}\}$ on $\mathcal{U}_1^1$, where V_1^t is the restriction of the operator V^t on $\mathcal{U}_1^1$. Since $\mathcal{U}_M^1 = \mathcal{U}_M \cap \mathcal{U}_1^1$ by construction, the operator B_1 is continuous by virtue of Theorem 4, and the semigroup $\{U^t : t \in \mathbb{R}_+\}$ is uniformly bounded, we obtain the inequality

$$\int_0^\tau \|V_1^t\|_{\mathcal{L}(\mathcal{U}_1^1;\mathcal{U}_M^1)}\, dt \leq \operatorname{const} \times \|B_1\|_{\mathcal{L}(\mathcal{U}_1;\mathcal{F}_1)} \int_0^\tau \|V_1^t\|_{\mathcal{L}(\mathcal{U}_1^1)}\, dt < \infty, \quad \tau \in \mathbb{R}_+. \tag{16}$$

According to Sobolev's inequality [12], the semigroup $\{W^t : t \in \mathbb{R}_+\}$ satisfies the estimate

$$\int_0^\tau \|W^t\|_{\mathcal{L}(\operatorname{dom} B_2; \overset{\circ}{W}{}_2^1(D))}\, dt < \infty. \tag{17}$$

Suppose that $\mathcal{U}_\alpha^1 = \mathcal{U}_\alpha \cap \mathcal{U}^1$, where $\mathcal{U}^1 = \mathcal{U}_1^1 \times \mathcal{U}_2$. Then inequalities (16) and (17) give the following lemma.

Lemma 3. *Under the conditions of Lemma 1, relation (11) takes place.*

Taking into account condition (12), we obtain the operator H as follows. The operator H is naturally represented as $H = H_1 \otimes H_2$, where $H_1 = A_{11}^{-1}(I - Q_0 - Q_1)F_1$, and $H_2 \equiv F_2$ (A_{11} is the restriction of the operator A_1 on $\mathcal{U}_1^1$). For the operator H, there is the statement that is similar to Theorem 7 for the operator F, i.e. $H \in \mathcal{C}^\infty(\mathcal{U}_M^1; \mathcal{U}_\alpha^1)$, where $\mathcal{U}_\alpha^1 = \mathcal{U}_\alpha \cap \mathcal{U}^1$.

Therefore, all the conditions of Theorem 2 are satisfied. Therefore, the following statement is valid.

Theorem 8. *Suppose that $\varkappa^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$. Then for any u_0 such that $u_0 \in \mathfrak{M}$, and some $T \in \mathbb{R}_+$, there exists the unique solution $u = (u_\sigma, 0, u_p, u_b)$ to problem (1), (2), which is a quasi-stationary trajectory, and $u(t) \in \mathfrak{M}$ for all $t \in (0, T)$.*

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НЕСТАЦИОНАРНАЯ МОДЕЛЬ НЕСЖИМАЕМОЙ ВЯЗКОУПРУГОЙ ЖИДКОСТИ КЕЛЬВИНА – ФОЙГТА НЕНУЛЕВОГО ПОРЯДКА В МАГНИТНОМ ПОЛЕ ЗЕМЛИ

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В работе исследуется задача Коши – Дирихле для системы уравнений Осколкова ненулевого порядка. Рассматриваемая математическая модель описывает течение несжимаемой вязкоупругой жидкости Кельвина – Фойгта в магнитном поле Земли. При этом учитывается, что на жидкость оказывают влияние различные внешние воздействия, зависящие как от координаты точки в пространстве, так и от времени. В первой части работы излагаются известные результаты, полученные авторами ранее, из теории разрешимости задачи Коши для полулинейных неавтономных уравнений соболевского типа. Во второй части проводится редукция рассматриваемой математической модели к указанной абстрактной задаче Коши. В третьей части доказывается основной результат – теорема существования и единственности решения. Находятся условия существования квазистационарных полутраекторий, а также описывается расширенное фазовое пространство исследуемой модели. Представленные в статье исследования обобщают результаты авторов для системы Осколкова, моделирующей движение вязкоупругой несжимаемой жидкости Кельвина – Фойгта нулевого порядка в магнитном поле Земли.

Ключевые слова: магнитогидродинамика; уравнения соболевского типа; расширенное фазовое пространство; несжимаемая вязкоупругая жидкость.

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